Analysis and Comparison of Two Types of Square-Wave Inverter Drives

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Abstract—The application of adjustable frequency static inverters to induction motor speed control has resulted in a diversity of inverter designs. In this paper, the state-variable formulation is applied to the analysis of two widely used drive systems incorporating square-wave inverters with 180° and 120° firing logic. Measured characteristics of an actual drive system are included, and the results are compared to an analytical solution. Steady-state speed-torque curves for each inverter supplying three markedly different types in induction motors are given. Performance of the two inverter drives are compared by a thorough investigation of typical operating points.

INTRODUCTION

As customer acceptance of static inverter-motor drives increases, a variety of inverter designs are evolving to satisfy the diverse needs of industry. When power requirements are modest, and very low speed operation is of no interest, the square-wave voltage inverter operating from an adjustable dc voltage bus continues to have advantages over more sophisticated and, hence, more complex and costly pulswidth modulated designs. Square-wave inverters can be categorized according to a number of criteria including type of commutating circuit, single or multiple grouping, type of dc power source, fixed or adjustable frequency, etc. An important element from the point of view of system performance is the permissible conduction period of the main thyristors. Although this conduction period can be fixed at any interval ranging from less than 90° to 180°, it is particularly advantageous to fix the interval at 120° or 180°. Both 120° and the 180° square-wave inverters are in wide use.

When utilizing the 180° mode of operation, three thyristors are always gated at any instant of time. This type of gating assures continuous conduction of ac current in the motor lines. In general, turn-off and turn-on signals are applied nearly simultaneously to corresponding top and bottom leg thyristors. Timing of the gate pulses to the main and auxiliary thyristors must be carefully adjusted so as to avoid shoot-throughs. Because of the critical timing of the inverter gating, the logic design for the 180° inverter is relatively complex. Analysis techniques for the 180° square-wave inverter drive have been developed, and its performance characteristics have been analyzed in detail [8]-[12].

When the 120° mode of operation is employed, only two thyristors are gated at any instant. Logic and gate pulse circuit design can, in this case, be simplified since a thyristor is not allowed to conduct until 60° after a turn-off signal is given to the complementary thyristor in the opposite half of a given phase. However, in this case, the ac line currents become discontinuous for high power factor loads. That is to say, an ac line current cannot reverse direction until a gating signal is applied to the thyristor with the proper polarity. During the zero current interval, the phase voltage appearing across the motor is not zero but is a back EMF arising from mutual coupling of the winding to both the rotor circuits and the other stator phases. Hence, for the 120° gating mode of operation, the inverter output voltage is a function of motor load.

The analysis of this type of inverter presents considerable difficulty since the resulting motor phase voltage is not a uniquely defined function of time. Sato [13] has developed a method of analysis for uncoupled static loads. Unfortunately, this type of analysis cannot be readily extended to the case of a motor load since a coupled active load introduces considerable complication into the analysis.

The problem of an inverter with 120° gating connected to a motor load has been approached by Sabbagh and Shewan [14] using the instantaneous symmetrical component transformation of Lyon [15]. In their analysis, the dc voltage across the inverter was assumed constant. It has been shown that the effects of the dc filter have a predominant effect on system performance especially in the lower frequency range [11], [12], [16], and it would be of interest to modify the analysis technique to include such effects. However, because of the extensive use of complex domain, time-varying phasors, lengthy algebraic manipulations are required and extension of their method to more complicated system configurations is difficult.

In this paper, an alternative approach to the problem employing the state-variable method [17], [18] is developed. The steady-state performance of an entire system including the effects of source impedance is obtained. Since the approach uses time-domain techniques, the solution obtained is the exact time response. Hence, the tedious infinite series frequency-domain techniques are entirely avoided. The differential equations that are derived are general in nature and can, if desired, be applied with equal validity to constant speed transients as well as steady-state.
state analyses. Also, the solution for an inverter drive with conventional 180° gating is obtained as a simplification of the solution for the 120° gating scheme. This solution is also significant in that it represents a departure from more conventional approaches to the solution of this problem. If necessary, the method can be readily extended to include additional effects such as saliency, multimotor loads, flexible shafts, and feedback control schemes.

BASIC ASSUMPTIONS

A simplified diagram of the system considered in this paper is given in Fig. 1. The drive system consists of an adjustable dc power source, a filter network, a three-phase inverter, and an induction motor. The dc power source shown as an adjustable dc voltage is generally a phase-controlled rectifier or dc chopper. A single LC filter has been assumed although many other configurations are possible and can be incorporated equally well into the analysis. The inverter is operated in the square-wave mode with either 120° or 180° gating.

In this paper, the following assumptions are made.

1) The dc power source may be considered as an ideal adjustable voltage having zero impedance. If necessary, the effects of source impedance can readily be incorporated by adjusting the impedance of the filter inductance L [19].

2) Harmonics arising from the bridge or chopper comprising the dc source are neglected.

3) The inverter is considered as a zero-impedance instantaneous switching device. Losses in the inverter can, if necessary, be accounted for by addition of an equivalent resistor in parallel with the filter capacitor C.

4) The induction machine is an idealized machine in which the stator and rotor windings are distributed so as always to produce a single sinusoidal space distribution of MMF in the air gap.

5) All parameters of the machine are assumed to be constant, and saturation of the magnetic circuit is neglected.

6) The system is in the steady state. In particular, the rotor speed is assumed to be constant. Although harmonic torques resulting from supply harmonics tend to induce speed oscillations [12], it is assumed that the rotor inertia is sufficiently large so as to minimize this effect.

7) Since open circuits occur only for highly resistive loads, they do not occur in the regenerative or braking mode of operation. Hence, only motor action will be considered.

SYMmetry CONDITIONS

Because of the symmetrical nature of the applied voltages and the resulting symmetry of the line currents, a considerable reduction is possible in the computing effort needed to obtain a solution. In order to illustrate this symmetry, a typical steady-state motor current is given in Fig. 2 corresponding to 120° gating. A diagram illustrating the thyristor gating sequence and current flow in the inverter for the assumed current condition is also shown. It can be noted that open circuits occur twice per phase per cycle provided that the current intersects the zero axis between $0 < \omega d < \pi/3$; otherwise the current is continuous. In this analysis, it will be assumed that open circuits occur. The solution for continuous current mode of operation is readily deduced once a solution has been obtained.

Since the inverter is triggered symmetrically, it is clear that three stator currents are both three-phase and half-wave symmetric. Hence, the zero current interval $\pi/3 - \varphi$ and phase angle $\varphi$ between voltage and current zero crossings are identical in all three phases. Because of this symmetry, it has been shown that only 1/6 of a complete cycle need be solved in detail, and a solution over this interval completely defines all variables over an entire cycle [18]. The time interval $T$ chosen for analysis is the 60° period initiated by the current $i_{in}$ reaching zero from the negative side. During the interval $T$, it is clear that two distinct circuit states exist: 1) as phase motor winding disconnected from the inverter, $b$s and $c$s connected as shown in
Fig. 3(a) and 2) all three motor phases connected to the inverter as shown in Fig. 3(b). These two circuit connections are defined as system states 2 and 1, respectively.

**STATE- VARIABLE ANALYSIS**

In the analysis of problems involving induction machinery, it has proven useful to transform the equations that describe the behavior of the machine to \( d-q \) axes fixed either on the stator or the rotor or rotating in synchronism with the applied voltages. In cases where open circuits occur in the stator, a stationary reference frame or one fixed in the stator is preferred. In the notation of Krause and Thomas [20], these equations, expressed in per unit, are given in matrix form by [21]

\[
\begin{bmatrix}
\begin{array}{ccc}
0 & r_s + \frac{p}{\omega_b} x_s & 0 \\
- \frac{p}{\omega_b} x_m & 0 & r_s + \frac{p}{\omega_b} x_s \\
0 & - \frac{p}{\omega_b} x_m & 0 \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
v_{qs}' \\
v_{ds}' \\
v_{r}' \\
\end{bmatrix}
= \begin{bmatrix}
r_s \frac{p}{\omega_b} x_m & 0 \\
- \frac{p}{\omega_b} x_m & r_s \frac{p}{\omega_b} x_m \\
- \frac{p}{\omega_b} x_m & r_s \frac{p}{\omega_b} x_m \\
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
i_{qs}' \\
i_{ds}' \\
i_{r}' \\
\end{array}
\end{bmatrix}
\times \begin{bmatrix}
\begin{array}{c}
v_{qs}' \\
v_{ds}' \\
v_{r}' \\
\end{array}
\end{bmatrix}
\]

or in the corresponding vector-matrix form as

\[\mathbf{v} = \mathbf{X} \frac{p}{\omega_b} \mathbf{i} + \mathbf{R} \mathbf{i}.\]  

(2)

In these equations, the superscript \( s \) is employed to denote that the \( d-q \) axes have been fixed in the stator. A \( p \) denotes the operator \( d/dt \). Although six equations are generally required to completely define the machine response, the two zero-sequence equations have been omitted since the sum of stator as well as rotor currents are zero. Also in (1), \( f_r \), termed the frequency ratio, is defined as \( \omega_s/\omega_b \), where \( \omega_s \) is the fundamental inverter electrical angular velocity and \( \omega_b \) is the base electrical angular velocity used to obtain the per-unit machine parameters. The quantity \( S \) is the slip defined as \( (\omega_s - \omega_r)/\omega_s \), where \( \omega_r \) is the equivalent electrical angular velocity of the rotor.

The voltages \( v_{qs}' \) and \( v_{qs}' \) are an equivalent set of voltages related to the phase voltages by the equations

\[v_{qs}' = v_{qs},\]  

(3)

\[v_{ds}' = \frac{1}{\sqrt{3}} (v_{qs} - v_{bs}).\]  

(4)

In order to obtain (3), it has been noted that the sum of the stator line-to-neutral voltages of a three-wire wye-connected machine is zero [20].

The \( d-q \) stator currents are similarly related to the stator phase currents by

\[i_{qs}' = i_{as},\]  

(5)

and

\[i_{ds}' = \frac{1}{\sqrt{3}} (i_{es} - i_{bs}).\]  

(6)

The stator referred \( ds-qs \) rotor currents are expressed in terms of the rotor phase currents by

\[i_{qs}' = \frac{N_r}{N_s} \left[ i_{sr} \cos \omega_d t + \frac{1}{\sqrt{3}} (i_{es} - i_{br}) \sin \omega_d t \right].\]  

(7)

or

\[i_{qs}' = \frac{N_r}{N_s} \left[ -i_{sr} \sin \omega_d t + \frac{1}{\sqrt{3}} (i_{es} - i_{br}) \cos \omega_d t \right].\]  

(8)

In order to obtain (5)–(8), it is again necessary to assume that the sum of the stator currents and the sum of the rotor currents are zero. In (7) and (8), \( N_r/N_s \) is the effective rotor-to-stator turns ratio and \( \omega_d \) denotes the relative displacement in electrical radians of the \( \theta \) rotor axis with respect to the \( \alpha \) axis, which have been assumed aligned at \( t = 0 \).

When peak rated line-to-neutral voltage and peak rated line current are chosen as base quantities, the electromagnetic torque expressed in terms of the \( ds-qs \) variables is

\[T_s = x_m (i_{qs}' i_{qs}' - i_{ds}' i_{ds}').\]  

(9)

The equations describing the behavior of the filter are given in per unit as

\[v_R = \left( r_L + \frac{p}{\omega_b} x_L \right) i_R + v_r,\]  

(10)

\[i_R = i_t + \frac{1}{x_C \omega_b} v_C,\]  

(11)

\[v_C = v_r + r_C (i_t - i_t),\]  

(12)

where \( x_L = \omega_b L \) and \( x_C = 1/\omega_b C \).
SYSTEM STATE 1—*iₘ*, *iₙ*, AND *iₑ* NONZERO

\[ \pi/3 < \omega \cdot \Delta < \varphi + \pi/3 \]

Although state 1 applies during the second part of interval \( T \), it is convenient to study this interval first. In state 1, all three lines are connected to the inverter as shown in Fig. 3(b). Equations describing the circuit connection are given by

\[ vₘₚ - vₘₖ = v_I \]
\[ \tau_c = \tau_v \]
\[ i_I = i_{es} \cdot \tau \]

Again utilizing the fact that the sum of the three stator phase voltages as well as the three stator currents add to zero for a three-wire system together with (3)–(6) relating phase to \( d-q \) variables, it can be shown that the \( d-q \) variables are expressed in terms of the inverter voltage and current by

\[ v_{es} = \frac{2}{3} v_I \]  
\[ v_{ds} = 0 \]  
\[ i_I = i_{es} \cdot \tau \]

From (1) and (10)–(12) together with (16)–(18), a single matrix differential completely defining system behavior in state 1 can be derived as

\[
\begin{bmatrix}
0 & r_s + \frac{2}{3} r_c + \frac{p}{\omega_b} x_s & 0 & \frac{p}{\omega_b} x_m & 0 & -\frac{2}{3} & -\frac{2}{3} r_c & i_{es} \\
0 & 0 & r_s + \frac{p}{\omega_b} x_s & 0 & \frac{p}{\omega_b} x_m & 0 & 0 & i_{ds} \\
0 & \frac{p}{\omega_b} x_m & -f_R(1 - S)x_m & r_{s}^\prime + \frac{p}{\omega_b} x_{s}^\prime & -f_R(1 - S)x_{s}^\prime & 0 & 0 & i_{qs} \\
0 & f_R(1 - S)x_m & \frac{p}{\omega_b} x_m & f_R(1 - S)x_{s}^\prime & r_{s}^\prime + \frac{p}{\omega_b} x_{s}^\prime & 0 & 0 & i_{qr} \\
0 & 1 & 0 & 0 & 0 & -\frac{1}{\omega_b x_c} & -1 & v_c \\
0 & -r_c & 0 & 0 & 0 & 1 & r_L + r_c + \frac{p}{\omega_b} x_L & i_{ig} \\
\end{bmatrix} \times \]

\[ \cdot \]

In compressed form, (19) can be written

\[ u = \left( \frac{p}{\omega_b} X_1 + R_1 \right) x \]

where \( X_1 \) is given by

\[ x = [i_{es}, i_{ds}, i_{qs}, i_{qr}, v_c, v_B]^T \]

\[ u = [0, 0, 0, 0, v_B]^T \]

\[ X_1 = \begin{bmatrix}
x_s & 0 & x_m & 0 & 0 & 0 \\
0 & x_s & 0 & x_m & 0 & 0 \\
x_m & 0 & x_{s}^\prime & 0 & 0 & 0 \\
0 & x_m & 0 & x_{s}^\prime & 0 & 0 \\
0 & 0 & 0 & 0 & 1/x_c & 0 \\
0 & 0 & 0 & 0 & 0 & x_L \\
\end{bmatrix} \]
In these equations, the subscript 1 has been appended to note those terms valid only for state 1. A T denotes the transpose. It can be noted that the elements of the augmented $R$ and $X$ matrices no longer necessarily have the units of ohms.

In standard state-variable form, (20) can be written

$$\frac{p}{\omega_b} x = A_1 x + B_1 u \quad (25)$$

where

$$A_1 = -X_1^{-1} R_1 \quad (26)$$

$$B_1 = X_1^{-1} \quad (27)$$

SYSTEM STATE 2—$i_{as} = 0$, $\varphi < \omega_b \phi < \pi / 3$

When stator current $i_{as}$ reaches zero, diode $D_1$ no longer conducts. Current will not flow in the $a$ phase motor winding until thyristor $T_1$ has been gated on. The connection of the circuit valid for this time interval is given in Fig. 3(a).

When current $i_{as}$ is zero, the voltage that appears across the $a$ phase is an induced EMF owing to mutual coupling with the other stator and rotor phases. By virtue of (5),

When $i_{as} = 0$, then $i_{qs} = 0$, and since $v_{as} = v_{qs}^*$, the open circuit voltage may be expressed in the $ds$-$qs$ axes as

$$v_{qs}^* = \frac{p}{\omega_b} x_m i_{qs}^* \quad (28)$$

From Fig. 3(a), the equations defining the remaining two winding voltages are

$$v_{ds} - v_{ka} = v_r \quad (29)$$

Thus, from (4)

$$v_{ds} = \frac{v_r}{\sqrt{3}} \quad (30)$$

Also, from Fig. 3(a)

$$i_r = i_{as} \quad (31)$$

Hence, in terms of $ds$-$qs$ variables

$$i_r = -\frac{1}{2} i_{qs}^* + \frac{\sqrt{3}}{2} i_{ds}^* \quad (32)$$

Substituting constraints (28), (30), and (32) into the general equations (1), (10), (11), and (12), the matrix differential equation describing the system in state 2 is

$$[\begin{array}{ccccccc} r_s + \frac{p}{\omega_b} x_s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s + \frac{1}{2\sqrt{3}} r_c & -f_R(1 - S)x_m & 0 & \frac{p}{\omega_b} x_m & 0 & -\frac{r_c}{\sqrt{3}} \\ 0 & 0 & r_r' + \frac{p}{\omega_b} x_r & -f_R(1 - S)x_r & 0 & 0 & 0 \\ 0 & -f_R(1 - S)x_m & f_R(1 - S)x_m & r_r' + \frac{p}{\omega_b} x_r' & 0 & 0 & 0 \\ -f_R(1 - S)x_m & 0 & 0 & 0 & \frac{1}{\omega_b x_c} & -1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & r_L + r_c \end{array}] \times \begin{bmatrix} i_{qs}^* \\ \vdots \\ i_{ds}^* \end{bmatrix} = \begin{bmatrix} v_{qs}^* \\ \vdots \\ v_r \\ v_{as} \\ v_{ds} \\ v_{rs} \\ i_R \end{bmatrix} \quad (33)$$
In abbreviated per-unit form, (33) can be written
\[ u = \left( \frac{p}{\omega_b} X_2 + R_3 \right) x \] (34)
or in standard state-variable form as
\[ \frac{p}{\omega_b} x = A_2 x + B_2 u \] (35)
where \( A_2 = -X_2^{-1} R_3 \) and \( B_2 = X_2^{-1} \). The matrices, \( R_3 \) and \( X_2 \) are defined in a similar manner as \( R_1 \) and \( X_1 \).

**SOLUTION FOR TIME INTERVAL \( T \)**
\( \varphi < \omega_d < \varphi + \pi/3 \)

During the time interval \( \varphi < \omega_d < \varphi + \pi/3 \), (35) is valid for system state 2, applies. Since the input vector \( u \) is constant over the interval, this solution, valid for any instant during the interval, is
\[ x(\omega, \varphi) = \exp \left[ \frac{A_2(\omega_d - \varphi)}{f_R} \right] x(\varphi) + \lambda_2(\omega, \varphi) B_2 u \] (36)
where \( \lambda_2(\omega, \varphi) \) is defined as
\[ \lambda_2(\omega, \varphi) = \frac{1}{f_R} \int_{\varphi}^{\omega} \exp \left[ \frac{A_2(\tau - \varphi)}{f_R} \right] d(\omega, \tau) \] (37)
Equation (37) can be solved as
\[ \lambda_2(\omega, \varphi) = \left( \exp \left[ \frac{A_2(\omega - \varphi)}{f_R} \right] - I \right) A_2^{-1} \] (38)

During the time interval \( \pi/3 < \omega_d < \varphi + \pi/3 \), (25) is valid. The input vector \( u \) remains constant. The solution is
\[ x(\omega, \varphi) = \exp \left[ \frac{A_1(\omega_d - \pi/3)}{f_R} \right] x(\varphi + \pi/3) + \lambda_1(\omega_d, \pi/3) B_1 u \] (39)
where \( \lambda_1 \) is defined similarly to \( \lambda_2 \) and is expressed by the equation
\[ \lambda_1(\omega_d, \pi/3) = \left( \exp \left[ \frac{A_1(\pi/3 - \omega_d)}{f_R} \right] - I \right) A_1^{-1} \] (40)


Equations (36), (38), (39), and (40) properly describe behavior of the system over the interval \( \varphi < \omega_d < \varphi + \pi/3 \) and constitute the solution over this time period. However, since the initial condition vectors \( x(\varphi) \) and \( x(\pi/3) \) are as yet unknown, a means of obtaining a proper initial condition vector must be developed before the solution is completely defined.

Setting \( \omega_d = \pi/3 \) in (36) and (38) and \( \omega_d = \varphi + \pi/3 \) in (39) and (40)
\[ x(\pi/3) = \exp \left[ \frac{A_3(\pi/3 - \varphi)}{f_R} \right] x(\varphi) + \lambda_2(\pi/3, \varphi) B_2 u \] (41)
and
\[ x(\varphi + \pi/3) = \exp \left[ \frac{A_1(\varphi + \pi/3 - \varphi)}{f_R} \right] x(\pi/3) + \lambda_1(\varphi + \pi/3, \pi/3) B_1 u \] (42)

where
\[ \lambda_2(\pi/3, \varphi) = \left( \exp \left[ \frac{A_1(\pi/3 - \varphi)}{f_R} \right] - I \right) A_2^{-1} \] (43)
\[ \lambda_1(\varphi + \pi/3, \pi/3) = \left( \exp \left[ \frac{A_1(\varphi + \pi/3 - \varphi)}{f_R} \right] - I \right) A_1^{-1} \] (44)

Substituting (41) in (42) results in
\[ x(\varphi + \pi/3) = \exp \left[ \frac{A_1(\varphi + \pi/3 - \varphi)}{f_R} \right] x(\varphi) \]
\[ + \exp \left[ \frac{A_1(\varphi + \pi/3 - \varphi)}{f_R} \right] \lambda_2(\pi/3, \varphi) B_2 \]
\[ + \lambda_1(\varphi + \pi/3, \pi/3) B_1 u \] (45)

By virtue of (45), a relationship has been obtained that relates the solution at time instant \( \omega_d = \varphi \) to the solution \( \pi/3 \) rad later at \( \omega_d = \varphi + \pi/3 \). It has been shown that for any three-phase and half-wave symmetric three-wire network the corresponding \( ds-qs \) variables are also related by a symmetry matrix \( S \), which equates the \( ds-qs \) stator or rotor variables at any instant to these same variables \( \pi/3 \) seconds displaced in time. For example, this symmetry of the stator \( ds-qs \) currents is expressed by the set of equations
\[ i_{qs}(\omega_d + \pi/3) = \frac{1}{3} i_{qs}(\omega_d) + \frac{\sqrt{3}}{2} i_{ds}(\omega_d) \] (46)
\[ i_{ds}(\omega_d + \pi/3) = -\frac{\sqrt{3}}{2} i_{qs}(\omega_d) + \frac{1}{3} i_{ds}(\omega_d) \] (47)

A similar pair of equations applies for the \( ds-qs \) stator referred rotor currents.

In the case of the inverter drive, the variables \( v_d, v_q \) and \( i_d \) on the dc side of the inverter have a basic sixth harmonic variation [19]. Hence, the pertinent dc side variables are related by
\[ v_d(\omega_d + \pi/3) = v_d(\omega_d) \] (45)
\[ i_d(\omega_d + \pi/3) = i_d(\omega_d) \] (49)

It is clear that for the case of an inverter drive, the symmetry equation relating the stator vector \( x(\varphi + \pi/3) \) to \( x(\varphi) \) is given by the matrix equation
\[ x(\varphi + \pi/3) = Sx(\varphi) \] (50)
\[ x_1 u \]
\[ \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \]

Utilizing (45) and (51), the unknown initial condition vector \( x(\varphi) \) can now be solved as

\[ x(\varphi) = \left( S - \exp \left[ \frac{A_{2\varphi}}{f_R} \right] \exp \left[ \frac{A_1(\pi/3 - \varphi)}{f_R} \right] \right)^{-1} \]

\[ \begin{bmatrix} \exp \left[ \frac{A_{2\varphi}}{f_R} \right] x_1 \left( \frac{\pi}{3}, \varphi \right) B_1 + x_2 \left( \varphi + \frac{\pi}{3}, \frac{\pi}{3} \right) B_2 \end{bmatrix} u. \]  

COMPUTATIONAL ASPECTS

Equation (52) describes a relationship that must be satisfied for steady-state operation. If \( x(\varphi) \) is found, the solution for the 60° interval \( \varphi < \omega t < \pi/3 + \varphi \) is known by means of (36) and (39). The solution for the remainder of the interval is developed from the symmetry relationship (50), which is clearly valid, not only for \( \omega t = \varphi \), but for any arbitrary time instant.

It should be mentioned that although the solution has been completely defined in terms of basic motor and filter parameters, the solution is only implicitly known by means of the defined quantities \( A_1, A_2, B_1, B_2 \). In some cases it would be desirable to obtain a solution wherein the parameters appear explicitly. This in fact can be done by substituting explicitly for these matrices in (36) and (39). The resulting equations, however, are cumbersome so that the algebra is best done with the aid of the computer.

It has been assumed that when \( \omega t = \varphi \), then \( i_{\varphi} = i_{\pi/3} = 0 \). Hence, the first element \( x_1 \) of the initial condition vector \( x(\varphi) \) must be identically zero. Since the unknown quantity \( \varphi \) appears on both sides of (52), an iteration is needed to locate the value of \( \varphi \) for which \( x_1 \) is zero. Clearly, the required value of \( \varphi \) must be in the interval \( 0 < \varphi < \pi/3 \) for 120° mode operation to occur.

A procedure that has been used with good success is to calculate \( x(\varphi) \) for \( \varphi = 0 \) and \( \pi/3 \) using (52). Since \( x_1(0) \) and \( x_1(\pi/3) \) are normally of different sign, a linear interpolation is used to estimate the value of \( \varphi \) for which \( x_1 \) is zero. A simple Newton–Raphson iteration can be used to converge on the correct value of \( \varphi \). In practice, the function \( x_1(\varphi) \) for values of \( \varphi \), \( 0 < \varphi < \pi/3 \), is nearly linear and only two or three iterations are needed for satisfactory convergence.

Should \( x_1(0) \) and \( x_1(\pi/3) \) be of the same sign, it can be assumed that zero intersections occur and the system is not operating in the 120° mode but effectively in the 180° mode. The solution for the 180° mode of inverter operation is given in the Appendix. Although it is conceivable that multiple intersections could occur, this would imply multiple solutions. Since the system is linear, such occurrences would be unexpected. Although care was taken to detect multiple solutions, none were identified during the course of this study.

Although the solution of (52) appears formidable, computation can be very efficiently arranged. Two approaches can be used to arrive at a solution. The matrix exponential functions \( \exp \left[ \frac{A_1(\pi/3 - \varphi)}{f_R} \right] \) and \( \exp \left[ \frac{A_{2\varphi}}{f_R} \right] \) can be calculated. Techniques for computing this function by power series expansion are well known [22]. The integrals of the matrix exponential \( \lambda_1(\pi/3, \varphi) \) and \( \lambda_2(\varphi + \pi/3, \pi/3) \) can then be obtained from (43) and (44). This method, however, necessitates three time-consuming matrix inverse operations in order to obtain \( x(\varphi) \).

Alternatively, \( \lambda_1(\pi/3, \varphi) \) and \( \lambda_2(\varphi + \pi/3, \pi/3) \) can be derived directly. The power series form for the matrix exponential integral is

\[ \lambda(t, 0) = I + \frac{A}{2 - 1} + \frac{A^2}{3!} + \frac{A^3}{4!} + \cdots. \]  

\[ (53) \]

Computation of this function is clearly analogous to computation of the matrix exponential function. Having obtained \( \lambda_1(\pi/3, \varphi) \) and \( \lambda_2(\varphi + \pi/3, \pi/3) \), \( \exp \left[ \frac{A_1(\pi/3 - \varphi)}{f_R} \right] \) and \( \exp \left[ \frac{A_{2\varphi}}{f_R} \right] \) can be obtained from (43) and (44) as

\[ \exp \left[ \frac{A_1(\pi/3 - \varphi)}{f_R} \right] = \lambda_1 \left( \frac{\pi}{3}, \varphi \right) A_1 + I \]  

\[ (54) \]

\[ \exp \left[ \frac{A_{2\varphi}}{f_R} \right] = \lambda_2 \left( \varphi + \frac{\pi}{3}, \frac{\pi}{3} \right) A_2 + I. \]  

\[ (55) \]

Hence, with the second approach, only one matrix inverse is needed to find \( x(\varphi) \). This alternative method was used to compute the results described herein. Although considerable algebra is required to obtain a solution, matrix operations can be very efficiently performed with a digital computer. The time needed to compute one value of \( x(\varphi) \) was typically 3–4 s on a GE 600 series computer.

COMPARISON OF COMPUTED AND TESTED RESULTS

In order to verify the analysis, the equations that have been developed were implemented into a digital computer program. The solution obtained from a computer analysis was compared to the tested results of an actual system.
The induction machine used for purposes of comparison was a 1.5-hp 60-Hz 110-V four-pole machine. Using rated voltage and output power as base quantities, the per-unit parameters of the machine are: \( r_s = 0.0347 \), \( r_f = 0.08785 \), \( x_s = 1.0715 \), \( x_f = 1.0936 \), \( x_m = 1.028 \). The parameters of the dc link filter are: filter inductance \( L = 12.85 \) mH, series resistance \( r_L = 0.53 \) \( \Omega \), shunt capacitance \( C = 5500 \) \( \mu F \), and shunt resistance \( r_C = 0.05 \) \( \Omega \). Referred to the same base ohms used to find the per-unit machine parameters, the per-unit filter parameters are: \( x_L = 0.448 \), \( r_L = 0.049 \), \( x_C = 0.0446 \), and \( r_C = 0.0046 \).

Fig. 4 shows a comparison of the results of a digital computer solution with the actual system. Rated frequency was supplied to the motor and the dc link voltage adjusted to supply rated rms voltage to the motor. In this case, the required dc source voltage was \( V_D = 1.65 \) per unit. Also, for this operating point \( f_R = 1.0 \), \( S = 0.125 \), and the load torque \( T_L = 5.9 \) ft-lb. (1.34 per unit). Although not measured in the physical system, a number of other system variables obtained from the computer solution are also shown. Although the computer program was written to compute per-unit quantities, all variables have been converted to normal units for purposes of comparison. It can be noted that the line current for the actual and computer solutions are very similar. Effects due to the commutation circuit tend to "smooth" the phase voltage of the tested system. However, a good comparison is evident.

**COMPARISON OF SYSTEM PERFORMANCE**

It is of interest to compare the difference in performance of various types of motors when operating from these two basic types of inverter supplies. Three different induction motors were selected to reflect a wide variation in performance characteristics. In each case, the stator frame size and winding configuration were identical. However, three different rotor structures were designed so that rated load was achieved at slips of 0.0313, 0.0622, and 0.0866, corresponding to a typical low slip, medium slip, and high slip motor. All three motors were four-pole designs rated at 5 hp, 230 V, 60 Hz. The per-unit parameters of the three motors are given in Table I.

Torque-speed curves for the three motors at rated frequency have been plotted in Fig. 5. In deriving these curves, the dc link voltage \( v_L \) was held constant at 1.57 per unit, and the effects of the dc link filter eliminated. This dc voltage corresponds to rated ac fundamental across the motor for continuous current (180° square wave) operation. Curves drawn with a solid line correspond to motors supplied from a 180° square-wave inverter, whereas dashed curves denote a 120° square-wave inverter supply. It can be noted that the maximum devel-
TABLE II
Comparison of System Performance at Three Operating Points

<table>
<thead>
<tr>
<th>Torque</th>
<th>Slip</th>
<th>RMS Line Voltage</th>
<th>RMS Line Current</th>
<th>Stator % Loss</th>
<th>Rotor % Loss</th>
<th>Ph-Pk Torque Ripple</th>
<th>Avg. Thyristor Current / Cycle</th>
<th>Avg. Diode Current / Cycle</th>
<th>Instantaneous RMS Current at Commutation</th>
<th>Filter Capacitor Current</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>0.0147</td>
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<td>0.0359</td>
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<td>0.328</td>
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<td>0.1550</td>
<td>0.0801</td>
<td>0.273</td>
<td>0.8350</td>
<td>0.0689</td>
</tr>
</tbody>
</table>

180° square wave inverter

<table>
<thead>
<tr>
<th>Torque</th>
<th>Slip</th>
<th>RMS Line Voltage</th>
<th>RMS Line Current</th>
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</tbody>
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120° square wave inverter with voltage compensation

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<th>Torque</th>
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<th>RMS Line Voltage</th>
<th>RMS Line Current</th>
<th>Stator % Loss</th>
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120° square wave inverter without voltage compensation

The reduction in effective ac voltage due to open circuiting of the stator phases, can, of course, be compensated by an increase in dc supply voltage with load. However, since this compensation must be accomplished through a time constant fixed by the LC filter, it is clear that the transient loading capability of a motor is reduced when supplied by the 120° inverter.

In many typical applications, inverter drives operate under two distinct modes: constant torque operation at low speed wherein the dc bus voltage is adjusted to maintain a constant flux level in the machine, and constant horsepower operation at high speed during which the dc bus voltage has been reached and the voltage amplitude, rather than the volts per hertz supplied to the machine, is maintained constant. It is evident that during the constant horsepower mode of operation, a motor supplied from the 120° inverter will, for the same load, run at a higher slip and thus incur greater losses than the same motor driven from a 180° inverter supply.

In order to further compare motor operation with the two types of inverter gating, Table II was prepared itemizing a number of important system design parameters. Since low slip motors are typically used with inverter supplies, the low slip motor of Table I was selected for the comparison. Load points of 0.5, 1.0, and 1.5 per unit were investigated corresponding to operation at light load, rated load, and heavy load. In all cases, the line frequency was fixed at 1.0 per unit (60 Hz). The effects of the filter were again eliminated for purposes of this comparison so that \( v_0 \) was fixed at 1.57 per unit. Although the analysis technique described in this paper has utilized peak rated voltage and current to form the per-unit system, the results of the computer study have been expressed, in this table only, as a per unit of rated rms \( L-n \) voltage (132.8 V), rated rms current (9.36 A) together with rated power (3.73 kW).

The first three rows summarize the case when the motor is supplied from a square-wave inverter with 180° gating. It can be noted that for all three load conditions the fundamental line-neutral voltage is constant verifying that the voltage for this type of inverter is independent of load.

The second three rows itemize results when the motor is fed from an inverter with 120° gating logic. Operation in the constant torque region was analyzed, and the dc link voltage was adjusted to compensate for open circuits. Compensation was derived from a voltage regulator signal developed from the average value of the rectified phase voltages. This type of voltage regulator action is described by the equation

\[
v_{ave} = \frac{1}{2\pi} \int_0^{2\pi} \left[ |v_{ac}| + |v_{dc}| + |v_{as}| \right] d(\omega t).
\]

For the case of the 180° inverter with rated volts per hertz, it can be shown that \( v_{ave} = 2f_N \). Hence, the inverter voltage for this case has been adjusted so that \( v_{ave} \) as defined by (56) is maintained at 2.0 per unit.

A comparison of operation with the regulated 120° square-wave inverter to that of a 180° square wave reveals a number of interesting features. It can be noted that operation at a light load (0.5 per unit) is identical in both cases, indicating that open circuits do not occur at light load. However, at rated and heavy loads, the rms motor phase voltage has increased, indicating that open circuits now occur. An increase in rms current and motor losses is observed, whereas the average current requirements in the thyristors decrease. Although these results appear to be contradictory, it should be recalled that the maximum conduction time of the main thyristors is only 120° rather than 180°. Hence, although the maximum instantaneous current is greater for the case of the 120° inverter, the conduction time is reduced so that the average current per cycle flowing through each thyristor is smaller.

This increase in current is apparent by a comparison of the instantaneous current at commutation. It can be noted that the current at commutation is substantially increased for the case of the 120° inverter, implying the need for larger capacity commutating components and indicating higher inverter switching losses. In the last column, the
rms filter capacitor current is given. Since the rms capacitor current is greater at high loads, a capacitor of larger kVA rating will probably be required when using a 120° inverter.

In the final three rows of Table II are tabulated the results of the 120° inverter-motor operation for a case where ceiling voltage has been reached at rated frequency. These results together with rows 1–3 for the 180° inverter permit comparison of operation of the two inverters in the constant horsepower mode. Again operation at light load is the same as for the 180° inverter case. However, since the de link voltage is fixed, it can be noted that the effective ac voltage is reduced with load so that the average thyristor current now becomes substantially greater at high loads. Losses in the motor and in the inverter commutation circuit also increase due to operation at a higher slip. In general, some type of current regulator will be used to protect the inverter against excessive currents. However, it is clear that current limiting will be accompanied by an additional loss in torque capability.

Although Table II indicates results for a low slip motor only, similar results were compiled for both the medium and high slip motors of Table I. The conclusions drawn from Table II were found to be valid for any of the three motors of Table I.

CONCLUSION

In this paper, complete solutions for two popular types of square wave inverter drives have been developed. The analysis has enabled entire systems containing these two types of inverters to be directly compared for the first time. Since the solution is obtained in the time domain, the method is readily programmed on the digital computer. The use of conventional infinite series, symmetrical component concept is avoided entirely. In addition, the system differential equations are immediately applicable to transient solutions and can be readily extended to include additional complexities such as multi-inverter or multi-motor drives, torsional resonance effects, and feedback control techniques.

State-variable techniques have proven to be an important new tool in ac machine analysis. Although the demise of the symmetrical component method is not imminent, it is apparent that state-variable techniques provide the analyst with a much more powerful tool with which to analyze complex drive system behavior. State-variable techniques have proven to be particularly useful in the analysis of systems that undergo numerous modes or circuit connections during normal operation. Since this feature is particularly characteristic of inverters, it is expected that state-variable techniques will find increasing use in the analysis of modern static ac drive systems.

APPENDIX

When the inverter is gated in the 180° mode or when the inverter is controlled with 120° gating but the instantaneous phase of the line current with respect to the motor phase voltage is greater than 60° (i.e., φ > π/3), continuous current conduction occurs. The solution for any time during the interval π/3 < φ < 2π/3 is

\[
\mathbf{x}(\phi) = \exp \left[ \frac{A_1(\phi - \pi/3)}{f_R} \right] \mathbf{x} \left( \frac{\pi}{3} \right) + \lambda_1(\phi - \pi/3) \mathbf{B}_1 \mathbf{u}
\]

(57)

where \( \mathbf{x} \), \( \mathbf{u} \), \( A_1 \), and \( B_1 \) are defined by (21), (22), (26), and (27), respectively, and \( \lambda_1 \) is in this case given as

\[
\lambda_1(\phi - \pi/3) = \frac{1}{f_R} \left( \exp \left[ \frac{A_1(\phi - \pi/3)}{f_R} \right] - 1 \right) A_1^{-1}. \]

(58)

In this case, (57) and (58) apply over the entire interval \( \pi/3 \) rad.

At the end of the interval \( \phi = 2\pi/3 \) and \( \lambda_1(2\pi/3, \pi/3) \) are given by

\[
\mathbf{x} \left( \frac{2\pi}{3} \right) = \exp \left( \frac{A_1\pi}{f_R} \right) \mathbf{x} \left( \frac{\pi}{3} \right) + \lambda_1 \left( \frac{2\pi}{3}, \frac{\pi}{3} \right) \mathbf{B}_1 \mathbf{u}
\]

(59)

\[
\lambda_1 \left( \frac{2\pi}{3}, \frac{\pi}{3} \right) = \frac{1}{f_R} \left[ \exp \left( \frac{A_1\pi}{f_R} \right) - 1 \right] A_1^{-1}.
\]

(60)

Because of symmetry

\[
\mathbf{x} \left( \frac{2\pi}{3} \right) = S \mathbf{x} \left( \frac{\pi}{3} \right)
\]

(61)

where \( S \) is defined by (51). Utilizing (59) and (61), the unknown initial conditions can be solved as

\[
\mathbf{x} \left( \frac{\pi}{3} \right) = \left[ S - \exp \left( \frac{A_1\pi}{f_R} \right) \right]^{-1} \lambda_1 \left( \frac{2\pi}{3}, \frac{\pi}{3} \right) \mathbf{B}_1 \mathbf{u}.
\]

(62)

It should be noted that the required initial condition for steady-state response has, in this case, been obtained in closed form. The solution for 180° gating given by (57), (58), and (62) represents a more comprehensive but nonetheless a simpler and more straightforward solution to the problem than recent alternative approaches by Harashima and Uchida [23] and Charlton [24].

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REFERENCES

Thomas A. Lipo (M'64–SM'72) was born in Milwaukee, Wis., on February 1, 1938. He received the B.E.E. degree with honors and the M.S.E.E. degree from Marquette University, Milwaukee, in 1962 and 1964, respectively, and the Ph.D. degree in electrical engineering from the University of Wisconsin, Madison, in 1968.

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Dr. Lipo is a member ofEta Kappa Nu, Pi Mu Epsilon, Tau Beta Pi, and Sigma Xi.

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He is currently Manager—Advanced Drive Systems Unit, Solid-State Power Control Branch of the Physics and Electrical Engineering Laboratory, Research and Development Center, General Electric Company, Schenectady, N. Y., where he heads a research program for the advanced development and application of adjustable speed drives. His work since he joined the Research and Development Center in 1959 has included the design and development of circuits utilizing the silicon-controlled rectifier and power transistor in static inverters, converters, cycloconverters, and dc and ac motor drives. Before assuming his present position he held responsibility for the development of power semiconductor circuits. He has been project engineer or contributor to the following development programs: dc/ac inverter for vehicle applications, three-phase to three-phase high-frequency inverter, cycloconverter for solar transmitters, converter and lighting inverter for rapid-transit application, a high-speed compressor drive, commutatorless dc motors, static motor starters, a hermetic-compressor drive using gate turn-off switches, a static ac voltage regulator, and de servo motor drives. He has been awarded two U. S. patents and has contributed to the book Principles of Inverter Circuits.

Mr. Turnbull is a member of Sigma Xi.