Stability Analysis of a Symmetrical Induction Machine

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Abstract—**A stability study of an induction machine is performed by applying the root-locus criterion to the equations which describe the behavior of the machine during small displacements about a steady-state operating point. This investigation reveals that a symmetrical induction machine may become unstable at low speeds (low frequencies) even though balanced, constant amplitude, sinusoidal voltages are applied to the stator terminals. Regions of machine instability are established from the results of a digital computer study. The results of an analog computer study are included to illustrate the modes of operation which occur within these regions. Regions of instability for changes of system parameters are presented and discussed. Also, it is shown that the root-locus criterion can be utilized to identify regions of lightly damped operation.**

Introduction

THE SQUIRREL-CAGE induction machine is a relatively inexpensive and rugged machine which in the past suffered from the disadvantage that its speed was not easily adjustable. Recent improvements in the capabilities of controlled rectifiers and continued decreases in the cost of their manufacture has made adjustable frequency speed control of ac machines practical. In many applications it may be necessary to operate these drives at 5 or 10 percent of the rated speed of the machine. It has thus become necessary to formulate analytical methods which accurately predict system performance over a wide range of operating frequencies.

In earlier papers it has been shown that during low-frequency operation the reluctance-synchronous machine as well as the synchronous machine can exhibit continuous oscillations in speed and in some cases may even become unstable [1]–[3]. These regions of instability occur with balanced sinusoidal applied stator voltages which are independent of load current (zero impedance source). Hence instability of machines having asymmetrical rotor windings is not necessarily due to an interaction of the machine and the power source.

Rogers has shown that an induction motor is typically lightly damped at low frequencies when operating from a balanced set of sinusoidal voltages [4]. Also, recent studies have indicated that variable speed induction motor drives may become unstable due to interaction of the machine and the filter of the power source [5]. Although not widely recognized, the symmetrical induction machine may also become unstable over a range of frequencies even though the machine operates from a balanced, sinusoidal, zero impedance source. The stability analysis of the induction machine operating from such a variable frequency power source is the subject of this paper.

In the analysis presented, small-displacement equations valid for small excursions about an operating point are established from the machine equations expressed in the synchronously rotating reference frame. A stability study is then performed by computing the characteristic roots or eigenvalues of the linearized system differential equations. Regions of instability are established for a particular machine, and the influence on system stability due to changes in machine parameters is investigated. In addition, it is shown that when a set of machine parameters does not result in instability, calculation of the eigenvalues provides a means of rapidly establishing areas of minimum damping, thus alerting the system designer to the operating region of greatest concern.

Definitions and Basic Equations

An idealized symmetrical induction machine is generally defined as a machine having the following characteristics.

1) The air gap is uniform.
2) The magnetic circuit is linear.
3) Three identical stator windings are distributed so as to produce a single sinusoidal MMF wave rotating in the air gap when excited by balanced sinusoidal currents.
4) The rotor coils or bars are arranged so that the rotor MMF wave can also be considered a space sinusoid having the same number of poles as the MMF wave of the stator.

Although the effect of temperature and frequency upon the parameters of the machine could be incorporated, these effects will not be considered in this analysis. Consistent with these assumptions the following per-unit set of equations which describe an idealized two-pole symmetrical induction machine in the arbitrary reference frame can readily be established [6].

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$$\begin{bmatrix} v_{qs} \\ v_{ds} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_s + \frac{p}{\omega_b} X_s & \frac{\omega}{\omega_b} X_s & \frac{p}{\omega_b} X_m & \frac{\omega}{\omega_b} X_m \\ -\frac{\omega}{\omega_b} X_s & r_s + \frac{p}{\omega_b} X_s & -\frac{\omega}{\omega_b} X_m & \frac{p}{\omega_b} X_m \\ \frac{p}{\omega_b} X_m & \frac{\omega - \omega_r}{\omega_b} X_m & r_r' + \frac{p}{\omega_b} X_r' & \frac{\omega - \omega_r}{\omega_b} X_r' \\ -\frac{\omega - \omega_r}{\omega_b} X_m & \frac{p}{\omega_b} X_m & -\frac{\omega - \omega_r}{\omega_b} X_r' & r_r' + \frac{p}{\omega_b} X_r' \end{bmatrix} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qr}' \\ i_{dr}' \end{bmatrix}$$

$$X_s = X_{ls} + X_m \tag{2}$$

$$X_r' = X_{lr}' + X_m. \tag{3}$$

In (1)–(3), p is the operator d/dt; r_s the stator resistance; r_r' the rotor resistance (referred to the stator winding); X_{ls}, X_{lr}' and X_m are the stator and rotor leakage reactances and magnetizing reactance, respectively. The electrical angular velocity of the rotor is denoted by ω_r. The reference frame rotates at a specified but arbitrary angular velocity ω.

If the machine is excited by a balanced sinusoidal set of three-phase voltages of angular velocity ω_e, the q- and d-axis voltages v_{qs} and v_{ds} are sinusoids of frequency $\omega_e - \omega$ and in the steady state the four d–q-axis currents assume this same frequency. Since it will be necessary to perturb the solution about a steady-state operating condition, it is of interest to select the reference frame wherein the voltage and current variables assume constant (dc) values for steady-state conditions. It is clear that this result will occur only if $\omega - \omega_e = 0$. Hence setting ω equal to ω_e the preceding equations are referred to the synchronously rotating reference frame wherein a set of orthogonal axes rotate at the angular velocity of the applied voltages. Equation (1) becomes

$$\begin{bmatrix} v_{qs}^e \\ v_{ds}^e \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_s + \frac{p}{\omega_b} X_s & f_R X_s & \frac{p}{\omega_b} X_m & f_R X_m \\ -f_R X_s & r_s + \frac{p}{\omega_b} X_s & -f_R X_m & \frac{p}{\omega_b} X_m \\ \frac{p}{\omega_b} X_m & f_R S X_m & r_r' + \frac{p}{\omega_b} X_r' & f_R S X_r' \\ -f_R S X_m & \frac{p}{\omega_b} X_m & -f_R S X_r' & r_r' + \frac{p}{\omega_b} X_r' \end{bmatrix} \begin{bmatrix} i_{qs}^e \\ i_{ds}^e \\ i_{qr}'^e \\ i_{dr}'^e \end{bmatrix} \tag{4}$$

where

$$S = \frac{\omega_e - \omega_r}{\omega_e}. \tag{5}$$

The quantity

$$f_R = \frac{\omega_e}{\omega_b} \tag{6}$$

is called the frequency ratio and may also be interpreted as the applied frequency expressed in per unit. The subscript e denotes variables expressed in the synchronously rotating reference frame.

The electromagnetic torque expressed in per unit is

$$T_e = X_m(i_{qs}^e i_{dr}'^e - i_{ds}^e i_{qr}'^e). \tag{7}$$

In addition, the per-unit equation expressing the electromechanical behavior of the system is expressed

$$T_e = \frac{2H}{\omega_b} p\omega_r + T_L. \tag{8}$$

In (8) the inertia constant H is expressed in seconds and defined as the ratio of the stored kinetic energy at base mechanical speed to the base power. T_L is the per-unit load torque applied to the machine.

Development of the Linearized Matrix Equation

Equations (4), (7), and (8) comprise an equivalent set of five nonlinear differential equations which describe the behavior of the induction machine. If a complete solution of the dynamic behavior of the induction machine is desired, these equations must be solved in detail by means of an analog or digital computer. However, if the local stability of the system is to be established, the equations can be simplified considerably by linearizing them about a steady-state operating condition.

This method of small displacements has historically been fruitful in analysis of electric machinery [7]–[9]. Because the resulting equations are linear, any of the conventional techniques of automatic control may be used to establish system stability. For example, the Routh test [2] or the Nyquist stability criterion [10] could be employed. In this paper a method analogous to the root-locus method is used to establish system stability.

If all variables in (4), (7), and (8) are allowed to change by a small amount about a steady-state operating point, and if the terms which describe the steady-state mode of operation are eliminated, the resulting equations can be written [5]

$$\begin{bmatrix} \Delta v_{qs}^e \\ \Delta v_{ds}^e \\ 0 \\ 0 \\ \Delta T_L \end{bmatrix} = \begin{bmatrix} r_s + \frac{p}{\omega_b} X_s & f_R X_s & \frac{p}{\omega_b} X_m & f_R X_m & 0 \\ -f_R X_s & r_s + \frac{p}{\omega_b} X_s & -f_R X_m & \frac{p}{\omega_b} X_m & 0 \\ \frac{p}{\omega_b} X_m & f_R S_0 X_m & r_r' + \frac{p}{\omega_b} X_r' & f_R S_0 X_r' & -(X_m i_{ds0}^e + X_r' i_{dr0}'^e) \\ -f_R S_0 X_m & \frac{p}{\omega_b} X_m & -f_R S_0 X_r' & r_r' + \frac{p}{\omega_b} X_r' & (X_m i_{qs0}^e + X_r' i_{qr0}'^e) \\ X_m i_{dr0}'^e & -X_m i_{qr0}'^e & -X_m i_{ds0}^e & X_m i_{qs0}^e & -2Hp \end{bmatrix} \begin{bmatrix} \Delta i_{qs}^e \\ \Delta i_{ds}^e \\ \Delta i_{qr}'^e \\ \Delta i_{dr}'^e \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} \quad (9)$$

where

$$S_0 = \frac{\omega_e - \omega_{r0}}{\omega_e}. \quad (10)$$

The zero subscripted quantities denote steady-state or operating point quantities.

Equation (9) can be written more concisely in partitioned matrix notation as [11]

$$\begin{bmatrix} \Delta \boldsymbol{v} \\ \Delta T_L \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{v}_{10} \\ \boldsymbol{v}_{20}^T & 0 \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{i} \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} + \frac{p}{\omega_b} \begin{bmatrix} \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0}^T & -2H\omega_b \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{i} \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} \quad (11)$$

where

$$\Delta \boldsymbol{v}^T = [\Delta v_{qs}^e, \Delta v_{ds}^e, 0, 0] \quad (12)$$

$$\Delta \boldsymbol{i}^T = [\Delta i_{qs}^e, \Delta i_{ds}^e, \Delta i_{qr}'^e, \Delta i_{dr}'^e] \quad (13)$$

$$\Delta \boldsymbol{v}_{10}^T = [0, 0, -(X_m i_{ds0}^e + X_r i_{dr0}'^e), X_m i_{qs0}^e + X_r' i_{qr0}'^e] \quad (14)$$

$$\Delta \boldsymbol{v}_{20}^T = [X_m i_{dr0}'^e, -X_m i_{qr0}'^e, -X_m i_{ds0}^e, X_m i_{qs0}^e] \quad (15)$$

$$\boldsymbol{R} = \begin{bmatrix} r_s & f_R X_s & 0 & f_R X_m \\ -f_R X_s & r_s & -f_R X_m & 0 \\ 0 & f_R S_0 X_m & r_r' & f_R S_0 X_r' \\ -f_R S_0 X_m & 0 & -f_r S_0 X_r' & r_r' \end{bmatrix} \quad (16)$$

$$\boldsymbol{X} = \begin{bmatrix} X_s & 0 & X_m & 0 \\ 0 & X_s & 0 & X_m \\ X_m & 0 & X_r' & 0 \\ 0 & X_m & 0 & X_r' \end{bmatrix} \quad (17)$$

and $\boldsymbol{0}$ represents a 4×1 column vector of zeros. In (12)–(15) a boldface italic letter is used to indicate a matrix or vector quantity. The superscript T denotes the transpose. The column vectors $\boldsymbol{v}_{10}$ and $\boldsymbol{v}_{20}$ denote quantities expressed in per-unit voltage and are established from steady-state operating conditions. Upon solving for the vector denoting the time derivatives of currents, (11) can be expressed

$$\frac{p}{\omega_b} \begin{bmatrix} \Delta \boldsymbol{i} \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{X}^{-1}\boldsymbol{R} & -\boldsymbol{X}^{-1}\boldsymbol{v}_{10} \\ \frac{1}{2H\omega_b} \boldsymbol{v}_{20}^T & 0 \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{i} \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} + \begin{bmatrix} \boldsymbol{X}^{-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & -\frac{1}{2H\omega_b} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{v} \\ \Delta T_L \end{bmatrix}. \quad (18)$$

Equation (18) constitutes the vector-matrix differential equation of the linearized system. The column vector $[\Delta \boldsymbol{i}^T, \Delta\omega_r/\omega_b]^T$ is the state vector which contains the set of linearized system state variables familiar to modern control theorists. The column vector $[\Delta \boldsymbol{v}^T, \Delta T_L]^T$ represents the set of system forcing functions. If the forcing function vector is set equal to zero, the solution to (18) is

$$\begin{bmatrix} \Delta \boldsymbol{i} \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} = e^{\boldsymbol{A}\omega_b t} \begin{bmatrix} \Delta \boldsymbol{i}(0) \\ \frac{\Delta \omega_r(0)}{\omega_b} \end{bmatrix} \quad (19)$$

where

$$\omega_b \boldsymbol{A} = \omega_b \begin{bmatrix} -\boldsymbol{X}^{-1}\boldsymbol{R} & -\boldsymbol{X}^{-1}\boldsymbol{v}_{10} \\ \frac{1}{2H\omega_b} \boldsymbol{v}_{20}^T & 0 \end{bmatrix} \quad (20)$$

and $[\Delta \boldsymbol{i}(0)^T, \Delta\omega_r(0)/\omega_b]^T$ is an arbitrary set of initial conditions. The matrix exponential function $e^{\boldsymbol{A}t}$ represents the unforced response of the system and is called the fundamental or state transition matrix of the system. Local stability is assured if all elements of the transition matrix approach zero asymptotically as time approaches infinity. Asymptotic behavior of all elements of the matrix occurs, in turn, whenever all of the roots of the characteristic equation have negative real parts [12]. The roots of the characteristic equation are given by those values of $\lambda' = \lambda/\omega_b$ for which the determinant

$$\left| \lambda' \begin{bmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{X}^{-1}\boldsymbol{R} & \boldsymbol{X}^{-1}\boldsymbol{v}_{10} \\ -\frac{1}{2H\omega_b} \boldsymbol{v}_{20}^T & 0 \end{bmatrix} \right| = 0. \quad (21)$$

The parameter λ' which has been introduced into (21) serves to normalize the roots of the characteristic equation to the base frequency ω_b. The roots of (21) may be computed by solving for the roots of the resulting polynomial equation in λ', or they may be calculated directly by determining the eigenvalues of the matrix $\boldsymbol{A}$.

A variation of the parameter $1/2H$ in (21) generates contours of eigenvalues solutions on the λ' plane. These contours are identical to the curves obtained by the well-established root-locus method employed extensively in control engineering [13]. Although it is not clear from the form of (21), the quantity $1/2H$ corresponds to the conventional root-locus gain parameter K. It is shown in the Appendix that this equation may be written in an equivalent form whereby this correspondence is evident.

Stability Studies

Solution of the roots of (21) provides a simple means of predicting the behavior of an induction motor at any operating frequency and for any torque load. Roots of λ' or λ/ω_b with

TABLE I

Fig.	r_s	r_r'	X_{ls}	X_{lr}'	X_m	H	V_k	V_m
1–3	0.025	0.015	0.1	0.1	3.5	0.1	0.025	1.0
4	varied	0.015	0.1	0.1	3.5	0.1	varied	1.0
5	0.025	varied	0.1	0.1	3.5	0.1	0.025	1.0
6	0.025	0.015	varied	varied	3.5	0.1	0.025	1.0
7	0.025	0.015	0.1	0.1	varied	0.1	0.025	1.0
8	0.025	0.015	0.1	0.1	3.5	varied	0.025	1.0
9	0.025	0.015	0.1	0.1	3.5	0.1	0.025	varied
10	0.025	0.015	0.1	0.1	3.5	0.5	0.025	1.0

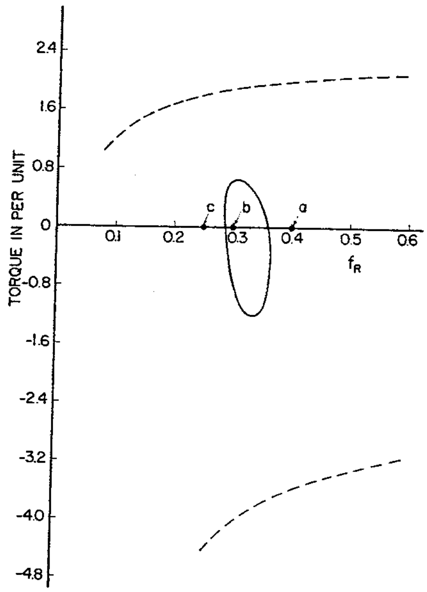

Fig. 1. Region of instability for symmetrical induction machine.

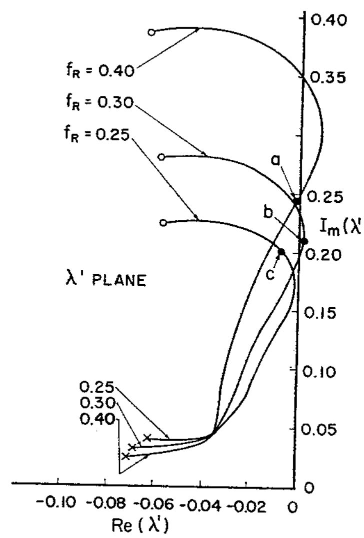

Fig. 2. Root-locus contours.

negative real parts correspond to terms which decrease exponentially with time. Roots of λ' having positive real parts result in elements in the state transition matrix which increase exponentially in time. In this study, 60 Hz is assumed to be rated frequency and the per-unit system employed is based on operation at this frequency ($\omega_b = 377$ rad/s). A change in steady-state operating speed (change in frequency) is easily incorporated by appropriate changes in the value of f_R.

In order to investigate thoroughly the stability of the induction machine, it is necessary to find the roots of the characteristic equation (21) for all practical operating frequencies and loading conditions and for a variety of practical machine parameters. Such a task can be accomplished quite readily by appropriate programming of a digital computer. Since root-locus contours typically involve repeated roots or pairs of roots with widely dissimilar real or imaginary parts, calculation of root-locus contours by computation of system eigenvalues could in the past only be accomplished with great difficulty. However, by means of the *Q–R* transform of Francis, eigenvalues of ill-conditioned matrices of large order can now be computed rapidly and accurately [14], [15].

In variable-speed systems, the amplitude of the applied voltages is typically decreased as frequency decreases in order to avoid saturation of the machine. However, if the voltage is decreased as a linear function of frequency, the breakdown torque is depleted significantly at low frequencies since an increased percentage of the applied voltage is dropped across the stator resistance as frequency is reduced. In this study, as a simple means of *iR* compensation the voltage required to produce rated flux linkage at rated load ($T_L = 1.0$ pu) and rated speed ($f_R = 1.0$) has been predetermined from steady-state considerations. For the reference set of parameters the terminal voltage required to satisfy this constraint is $V = 1.025$ pu. When operating from a variable frequency source, the terminal voltage has been adjusted so that for any frequency

$$V = V_k + f_R V_m \tag{22}$$

where $V_k = 0.025$ and $V_m = 1.0$ pu. From the form of (22) it is clear that the constant factor V_k serves to compensate for the stator *iR* drop. It can be noted that when $f_R = 1.0$, the terminal voltage $V = 1.025$.

A region of instability for a 60-Hz 7.5-hp 220-volt four-pole three-phase induction motor is shown in Fig. 1. The per-unit parameters of this machine are given in Table I. The terminal voltage has been adjusted in accordance with (22). The dashed lines shown in Fig. 1 indicate the breakdown or maximum steady-state torque at the various operating frequencies. Negative torque output denotes induction generator action.

The continuous contour shown in Fig. 1 forms the boundary between stable and unstable regions of operation. More specifically, this contour connects all operating points which yield a pair of complex-conjugate eigenvalues with a zero real part. The root-loci charts shown in Fig. 2 serve to demonstrate the significance of the closed contour given in Fig. 1. The three

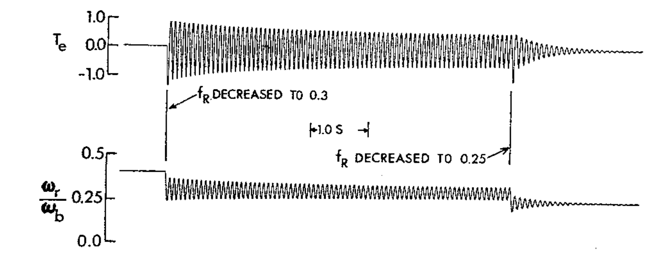

Fig. 3. Frequency switching, analog computer study.

root-locus plots in Fig. 2 correspond to initial operating points indicated in Fig. 1 as *a*, *b*, and *c*. These operating points occur at a constant loading condition of zero torque at frequencies of $f_R = 0.4$, 0.30, and 0.25, respectively. Since the frequency ratio f_R is defined as the ratio of applied frequency to base frequency, and since base frequency has been selected as 60 Hz, these operating points correspond to frequencies of operation of 24, 18, and 15 Hz.

With $f_R = 0.40$, in Fig. 2, the root-locus contour of the dominant pair of roots passes into the right-half plane indicating possible unstable points of operation. However, for the inertia being considered, the gain parameter $1/2H$ is 5.0, thus, point *a* on this locus for zero-load torque is in the left-half plane corresponding to a stable operating condition. Since the remaining roots contain large negative real parts, these roots have negligible effect upon system response. The location of these roots have not been included in Fig. 2. With $f_R = 0.30$ the root-locus contour again passes into the right-half plane. In addition, the gain parameter $1/2H = 5.0$ fixes operating point *b* in the right-half plane. Therefore, the system is unstable for zero-load torque with $f_R = 0.30$. When $f_R = 0.25$, the gain parameter $1/2H = 5.0$ locates the dominant root in the left-half plane (point *c*). That is, the induction motor is again stable for $f_R = 0.25$ and zero-load torque.

In Fig. 2 the complete locus of the dominant root is plotted for a given operating frequency and loading condition. This locus was obtained by varying the gain constant $1/2H$ from zero to infinity. However, it is clear that the entire root locus need not be calculated at each operating point in order to establish system stability since only one set of points on the locus corresponds to a specified inertia constant H.

In the foregoing stability analysis the equations which describe the dynamic performance of the induction machine have been linearized by the method of small displacements. This analysis reveals that regions of instability can occur over a range of frequencies. It is of interest to compare the results obtained from the linearized set of equations to an analog-computer simulation of the actual nonlinear set of system equations. The analog-computer traces of system variables for step changes in applied electrical frequency is shown in Fig. 3.

With the machine initially operating at zero torque and $f_R = 0.40$, the frequency ratio is suddenly switched to 0.30 (18 Hz). At the instant of switching the terminal voltage was also changed in accordance with (22). This switching corresponds to a change from point *a* in Fig. 1, a stable point of operation, to point *b* an unstable operating point. The analog-computer trace demonstrates the sustained oscillations which occur with operation inside the region of instability. After the switching transients had disappeared, the frequency ratio was then switched to point *c*, a stable operating point. Positive system damping is

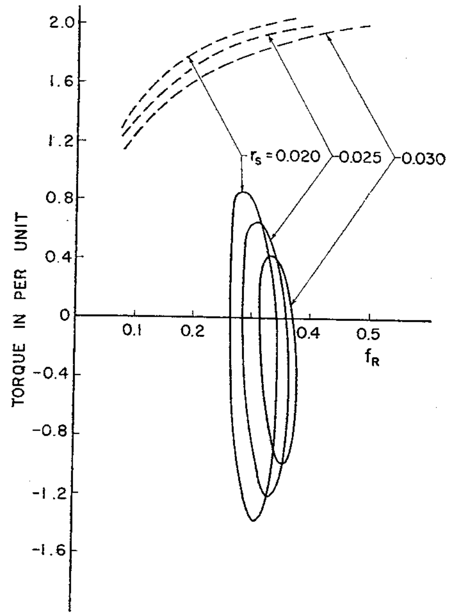

Fig. 4. Regions of instability for different values of stator resistance.

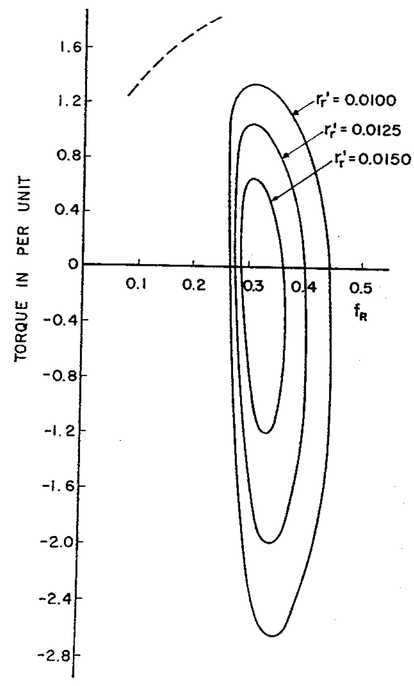

Fig. 5. Regions of instability for changes in rotor resistance.

evident at the stable operating point, and the induction machine again assumes operation at a constant rotor speed.

The results of the stability study based on the theory of small displacements suggests that sustained oscillations will occur only at operating points on the contour. Oscillations will be damped for operation outside the region of instability, and negative damping will exist in the unstable region. One might expect that operation at a point within the region of instability will cause unbounded increases in oscillation, causing the motor to reach its breakdown torque, subsequently resulting in a braking of the machine. However, positive damping occurs during that part of the oscillation which includes operating

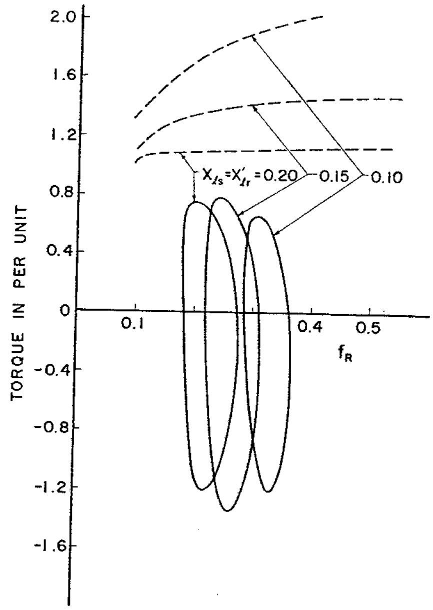

Fig. 6. Regions of instability for increase in stator and rotor leakage reactances.

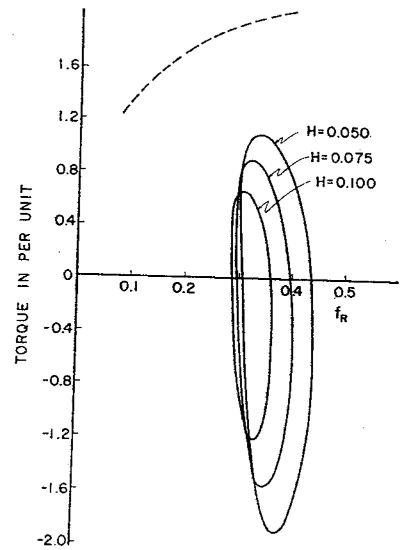

Fig. 8. Regions of instability for decrease in system inertia.

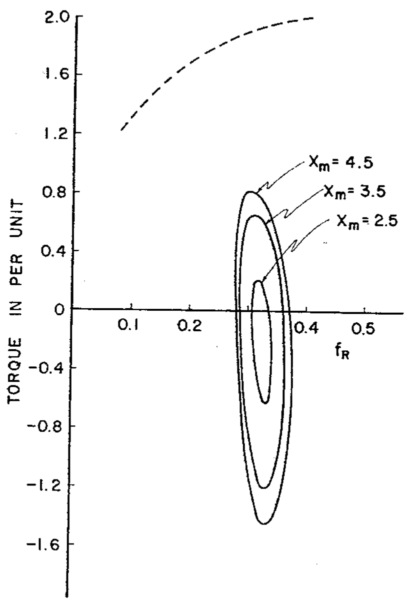

Fig. 7. Regions of instability for changes in magnetizing reactance.

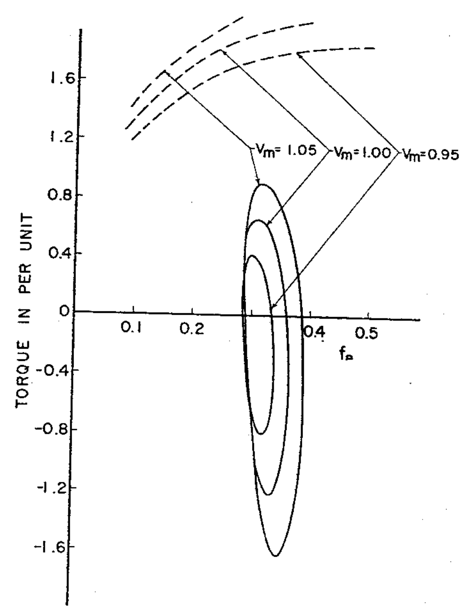

Fig. 9. Regions of instability for different values of applied stator voltage.

points outside the region of instability. Sustained oscillations will occur when the equivalent damping over a complete cycle is zero. It can be concluded from this study, that sustained oscillations of the machine variables typically occur rather than a breakdown of the machine.

The contours shown in Figs. 4–9 illustrate the effect of various machine and system parameters upon machine stability. In order to facilitate a direct comparison, the contour given in Fig. 1 is also included in these figures. The parameter values of the reference or base system and the quantities which were varied in each case are given in Table I.

Variation in the region of instability due to a change in stator resistance is given in Fig. 4. Regions of instability are shown for r_s = 0.020, 0.025, and 0.030 pu. For these curves, the constant V_k in (22) has been altered for each value of stator resistance so as to maintain rated flux linkage at rated frequency and rated load torque. It was found that an increase of stator resistance to r_s = 0.035 pu eliminates the region of instability. Hence the induction machine can be stabilized by an appropriate increase in stator resistance. This result is in contrast to results obtained in previous studies involving reluctance-synchronous and synchronous machines [1], [2].

The contours shown in Fig. 5 show the effect of rotor resistance on the instability region. In particular, the plot reveals that increasing rotor resistance tends to stabilize the machine which is in agreement with established theory [8].

Regions of instability resulting from changes in stator and rotor leakage reactances are displayed in Fig. 6. A change in

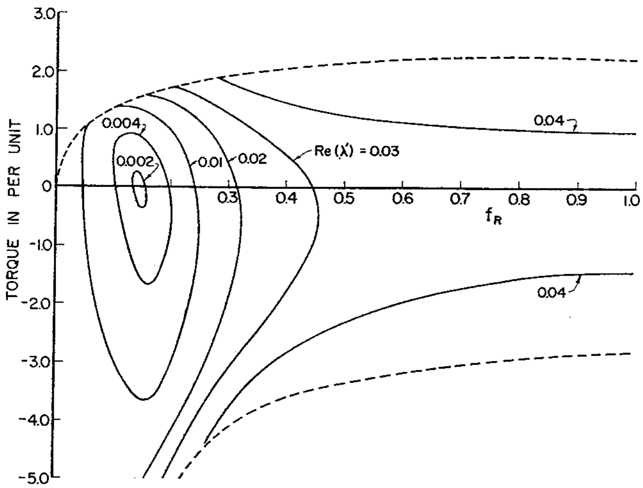

Fig. 10. Contours of constant damping.

both stator and rotor leakage reactances demonstrates the effects of varying the coupling of the stator and rotor circuits. It is interesting to note that an intermediate value of X_{ls} and $X_{lr}' = 0.15$ produces the largest region of instability. A decrease in leakage reactances from this value tends to decrease the region of instability, and when $X_{ls} = X_{lr}' = 0.05$ the machine is stable over all regions of operation. An increase from this value of leakage reactance also tends to decrease the region of instability; however, in this case the region of instability moves to the left (lower frequencies) and does not entirely disappear. Also, it may be noted that at a given frequency of operation, the machine can be stabilized for either an increase or decrease in leakage reactances. Hence at $f_R = 0.28$ and $X_{ls} = X_{lr}' = 0.15$ the machine is unstable, but for $X_{ls} = X_{lr}' = 0.10$ or 0.20 the machine is stable for all loading conditions.

The effects of an increase or decrease in magnetizing reactance is given in Fig. 7. The curves demonstrate that a small magnetizing reactance is desirable from the point of view of system stability.

Variation in the region of instability due to different system inertias is given in Fig. 8. Regions of instability are shown for $H = 0.05$, 0.075, and 0.1 second. The machine is stable over all regions of operation when $H = 0.15$. It is clear that an increase in inertia tends to promote system stability. It is interesting to note that at a fixed frequency an increase or decrease in system inertia may tend to stabilize the machine. For example, at $f_R = 0.3$ and $H = 0.1$ second the system is unstable, but when $H = 0.15$ or 0.05 the induction machine is stable for all loading conditions at this frequency.

In Fig. 9 the result of an increase in terminal voltage upon stability is shown. In the previous sets of curves, the terminal voltage has been decreased so as to maintain rated flux linkage within the machine for rated frequency and rated load torque. In Fig. 9 the terminal voltage constraint given by (22) has been modified so that $V_m = 1.05$ and 0.95 pu. The results of this study indicate that the region of instability increases with an increase in terminal voltage.

The majority of induction machines in current manufacture do not demonstrate the type of low-frequency instability studied in this paper. For example, machines of large horsepower rating typically have an inertia constant H of 0.5 second or greater. Hence this type of machine would not be expected to experience operating point instability. However, regardless of the parameters of the machine, regions of lightly damped operation may occur. Regions of small damping can rapidly be identified by calculating the eigenvalues of the linearized machine equations (21).

In the analysis presented, regions of instability have been identified for several choices of machine parameters. The contour which separates the stable and unstable regions corresponds to operating points for which the real part of the dominant pair of eigenvalues is zero. Contours of constant nonzero damping may readily be constructed in much the same manner. In Fig. 10 the inertia constant H of the reference machine has been set equal to 0.5 second. The contours given in Fig. 10 denote operating points for which the dominant pair of characteristic roots have the same real part. It is readily noted that minimum damping for this machine occurs at $f_R = 0.15$.

Conclusion

A method of determining the stability of an induction motor operating from a variable frequency power source has been set forth. The linearized equations describing the behavior of the machine about an operating point have been cast into a simple matrix formulation which permits the stability problem to be conveniently solved by matrix multiplication, matrix inversion, and matrix eigenvalue subroutines which are readily available in most computer libraries.

Although it is not widely recognized that instability of a symmetrical induction motor can occur, it has been demonstrated that while most induction machines are stable, regions of instability can exist if the parameters of the machine are incorrectly chosen. Instability is particularly likely to occur in machines of small horsepower rating for which the inertia constant of the machine is small. Attempts to reduce the stator or rotor i^2R losses or to reduce the magnetizing current all enhance the possibility for machine instability.

When machine instability does not occur, regions of lightly damped operation may exist in regions from 5 to 25 Hz. When the induction machine is furnishing mechanical power to a pulsating load such as a compressor or pump, it is of importance to establish the amount of system damping available from the induction machine. Computation of the eigenvalues of the system characteristic equation provides the system designer with valuable information with which to properly match a mechanical device to its power source.

It is important to note that the analysis contained in this paper may be readily extended to analyze the stability of other types of machinery. In particular, the matrix equation (18) can be easily modified to study the stability of a reluctance-synchronous or synchronous machine. Therefore, development of a computer program using the method of analysis set forth in this paper would provide a means of investigating the stability of a wide variety of ac machinery.

Appendix

The characteristic equation corresponding to the linearized set of induction motor equations is

$$\left| \lambda' \begin{bmatrix} I & 0 \\ 0^T & 1 \end{bmatrix} + \begin{bmatrix} X^{-1}R & X^{-1}v_{10} \\ -\frac{1}{2H\omega_b} v_{20}{}^T & 0 \end{bmatrix} \right| = 0. \tag{22}$$

Upon adding corresponding terms of the two partitioned matrices (22) becomes

$$\left|\begin{bmatrix} \lambda' I + X^{-1}R & X^{-1}v_{10} \\ -\frac{1}{2H\omega_b} v_{20}^T & \lambda' \end{bmatrix}\right| = 0. \tag{23}$$

If the determinant is expanded by minors along the second row

$$\lambda' |\lambda' I + X^{-1}R| + \frac{1}{2H}\left|\begin{bmatrix} \lambda' I + X^{-1}R & X^{-1}v_{10} \\ -\frac{1}{\omega_b} v_{20}^T & 0 \end{bmatrix}\right| = 0. \tag{24}$$

Equation (24) can be rearranged in the form

$$-1 = \frac{\frac{1}{2H}\left|\begin{bmatrix} \lambda' I + f_R\omega_b X^{-1}R & X^{-1}v_{10} \\ \frac{1}{\omega_b} v_{20}^T & 0 \end{bmatrix}\right|}{\lambda' |\lambda' I + X^{-1}R|}. \tag{25}$$

It is clear that (25) is equivalent to a polynomial expression of the form

$$-1 = K\frac{N(\lambda')}{D(\lambda')}. \tag{26}$$

In this equation $N(\lambda')$ is a fourth order polynomial in λ' and corresponds to the zeros of the open-loop transfer function; $D(\lambda')$ is a fifth order polynomial which yields the poles of the open-loop transfer function. The quantity $1/2H$ thus corresponds to the gain K of the open-loop system, and as $1/2H$ is varied from 0 to ∞, the root-locus contours of the overall closed-loop system are generated on the λ' plane.

References

[1] T. A. Lipo and P. C. Krause, "Stability analysis of a reluctance-synchronous machine," *IEEE Trans. Power Apparatus and Systems*, vol. PAS-86, pp. 825–834, July 1967.

[2] R. G. Hoft, "Liapunov stability analysis of reluctance motors," *IEEE Trans. Power Apparatus and Systems*, vol. PAS-87, pp. 1485–1491, June 1968.

[3] T. A. Lipo and P. C. Krause, "Stability analysis for variable frequency operation of synchronous machines," *IEEE Trans. Power Apparatus and Systems*, vol. PAS-87, pp. 227–234, January 1968.

[4] G. J. Rogers, "Linearised analysis of induction-motor transients," *Proc. IEE* (London), vol. 112, pp. 1917–1926, October 1965.

[5] T. A. Lipo and P. C. Krause, "Stability analysis of a rectifier-inverter induction motor drive," *IEEE Trans. Power Apparatus and Systems*, vol. PAS-88, pp. 55–66, January 1969.

[6] P. C. Krause and C. H. Thomas, "Simulation of symmetrical induction machinery," *IEEE Trans. Power Apparatus and Systems*, vol. PAS-84, pp. 1038–1053, November 1965.

[7] R. H. Park, "Two-reaction theory of synchronous machines, pt. II," *AIEE Trans.*, vol. 52, pp. 352–355, June 1933.

[8] C. Concordia, "Induction motor damping and synchronizing torques," *AIEE Trans. (Power Apparatus and Systems)*, vol. 71, pp. 364–366, 1952.

[9] J. W. Lynn, "Tensor analysis of electrical machine hunting," *Proc. IEE* (London), vol. 105, pt. C, pp. 420–431, 1958.

[10] A. S. Aldred and G. Shackshaft, "A frequency-response method for the predetermination of synchronous-machine stability," *Proc. IEE* (London), vol. 107, pt. C, pp. 2–10, 1960.

[11] V. N. Faddeeva, *Computational Methods of Linear Algebra.* New York: Dover, 1959.

[12] L. S. Pontryagin, *Ordinary Differential Equations.* Reading, Mass.: Addison-Wesley, 1962.

[13] C. A. Stapleton, "Root-locus study of synchronous-machine regulation," *Proc. IEE* (London), vol. 111, pp. 761–768, April 1964.

[14] J. G. Francis, "The *Q-R* transformation," *Computer J.*, vol. 4, pp. 265–271; 332–340, 1961.

[15] J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* London: Oxford University Press, 1965.

Discussion

S. D. T. Robertson and N. Ramesh (University of Toronto, Toronto 5, Ont., Canada): The authors are to be congratulated for a lucid presentation of the results of their analysis.

The equations for stability analysis for small displacements have been formulated quite some time back by Concordia [8] and Kron [16]. Variable frequency static drives have brought the problem into focus in recent times. The method of analysis as proposed by Kron takes two forms, namely, synchronising and damping torque method and control system techniques.

Rogers [4] has used the root-locus technique to show not only "light damping" effects but also the possibility of instability in induction machines when operating from low-frequency supplies. This paper also analyses the induction motor using the root-locus technique. However, its major contribution seems to be in the clarity of the results and conclusions.

What seems to be desirable are analytic conditions, which show the bifurcations in the parametric space, for various regions of stability, instability, etc. It should be noted that Lyapunov's second or direct method offers a possibility of obtaining such conditions. It would be interesting to know if the authors have actually observed sustained oscillations similar to those predicted in Fig. 3 in an actual motor.

Under saturated conditions it can be shown that the magnetising inductance decreases. In this case from Fig. 7 the stability would be improved. However, Fig. 9 shows an adverse effect on the performance when the supply voltage is increased. As the increase in supply voltage increases saturation it would seem by combining the results of Figs. 7 and 9 that in practice the two effects would tend to nullify each other. The authors' comments on this would be much appreciated.

Finally, the neglect of mechanical damping in the mechanical equation would give pessimistic results. Would the authors remark on the significance of this term?

References

[16] G. Kron, *Tensors for Circuits.* New York: Dover, 1959.

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Robert H. Nelson, Thomas A. Lipo, and Paul C. Krause: We wish to thank Prof. Robertson and Mr. Ramesh for their comments. The comment regarding terminal voltage and saturation is of interest. The effect of saturation was discussed in the closure of [1]. Also, applications of Lyapunov's methods are discussed in the discussion by Lipo and Krause and closure by Hoft [2]. Neglecting mechanical damping will give pessimistic results. Mechanical damping was included in Hoft's work [2].

Although we have not had the opportunity to build an actual variable frequency drive system, we have had several opportunities to compare our results with working systems. Perhaps the most vivid comparison regarding the observation of instability is presented in the discussion and closure of [5]. The comparison discussed here was quite favorable.

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