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Abstract—The scope of static ac drives is increasing rapidly, and with this increase has evolved the need to devise control strategies for a variety of new applications. Purely analytical approaches to design of control systems for ac drives are hindered by the highly coupled nature of the ac induction motor equations which, in the past, necessitated lengthy manual derivations of transfer functions. This paper presents an alternative approach using a state variable formulation. The equations are arranged in a form such that the entire derivation procedure can be relegated to a digital computer. Transfer function poles, zeros, and gain for any practical input-output pair of variables can be rapidly obtained.

INTRODUCTION

As applications for wide range, adjustable speed, static ac drives increase, these new applications result in a variety of performance criteria. Since control system structure is largely dependent on performance criteria, these new applications of ac equipment have generated a wide variety of system feedback configurations. For example, control of torque can be maintained by sensing a signal proportional to torque directly by means of a transducer or alternatively, indirectly by regulating stator current magnitude, air gap flux magnitude, or the power component of stator current, together with slip frequency. Moreover, these variables can be sensed either directly or indirectly by implicit means. Indirect sensing implies measurement of still other variables. An important aspect of system design is the proper adjustment of system feedback compensators. In view of the large number of possible feedback configurations which can be used to accomplish the same task, it is desirable to have available a fast, accurate technique to identify the transfer functions between specific motor inputs and outputs.

Traditional approaches to feedback design of drive systems has, to a large extent, relied on frequency response measurements on the actual system.¹ Feedback components are selected based on the measured gain and break frequencies at selected operating conditions. Proper instrumentation is, however, time consuming. Also, measurements can be taken only when the assembly of the drive is in its final stages. Hence, changes in the system necessitated by the control study are costly. More recently, an analog computer simulation has been utilized in the design of the feedback network.² Although this work can be carried out at an earlier stage, neither method promotes a thorough understanding of mechanisms which influence the linearized system transfer function.

The transfer function method together with root locus has, of course, long been a favorite control system design technique. Since this approach is understood by both control and motor designers, it forms an ideal basis by which to perform control studies. Recently, much attention has been directed to the computation of the transfer function poles (characteristic roots) of the linearized induction machine equations.³⁻⁵ Such information has proven useful in the analysis of motor instability which results when an ac motor is driven from an adjustable frequency supply. The system poles can be easily found by computing the eigenvalues of the system state matrix. Hence, open loop

motor stability problems can be investigated in a relatively simple manner.

It can be shown that the transfer function poles are independent of the input or output under consideration. The transfer function zeros, on the other hand, change with choice of input and output and must be derived independently for each input-output pair.

No simple means exists which permits computation of the transfer function zeros similar to that used for the poles. For the case of the ac machine, an attempt to derive a transfer function between a specific input (e.g., voltage amplitude) and output (e.g., flux amplitude) without simplifying assumptions leads to a tedious algebraic derivation. This clearly limits the usefulness of such an approach. Detailed transfer functions for certain specific inputs and outputs have been manually derived.^{3,6-10} However, if all possible alternatives are to be investigated during the course of a total system design, derivation of all of the required transfer functions purely by manual means is clearly impractical.

In view of this difficulty, motor transfer functions in the past were derived either neglecting electrical time constants entirely¹¹ or including a first order lag to account for the rotor transient time constant.¹² This approximation essentially assumes that the stator resistance is negligible compared to the motor leakage reactance. Transfer functions of this type produce adequate results for power system stability studies and have also been used in the analysis of ac motor control.^{13,14}

At normal supply frequencies, it has been determined that the dominant mode indeed corresponds to the rotor time constant.^{3,4} However, at low excitation frequencies, the dominant mode is influenced by stator resistance.³ An induction motor must, in general, be represented as a fifth order system when changes in speed are considered. Depending upon the location of the poles relative to the zeros, certain modes may or may not be neglected for a specific transfer function. It is clear that when evaluating the effect of feedback compensators, an exact knowledge of both zero and pole locations is essential.

The state variable approach has been demonstrated to be a superior alternative to more conventional methods of ac motor analysis in a number of applications.^{15,16} However, in most cases, the method has served merely as a convenient notational framework. Many inherent advantages of the state variable method have not been fully exploited. This paper presents a state variable approach to the derivation of induction motor transfer functions. The system, comprising the motor and source, is completely general in that a doubly-fed machine is considered with external source impedance in both the stator and rotor supply. It is shown that the system equations can be arranged in a form such that the proper defining equations for any motor transfer function can be readily obtained. A transformation of these equations can be made which permits the poles, zeros, and gain of the desired transfer function to be computed by standard root solving subroutines. The entire computation can be efficiently programmed on a digital computer. Transfer functions for any practical input and output can be rapidly obtained.

A study is conducted with sample induction motor parameters, and the results for a wide variety of input-output pairs tabulated. It is shown that transfer function zeros vary widely for choices in input and output. The effect of speed changes, normally neglected in such an analysis, is demonstrated to have an important effect on the character of many motor transfer functions. Also, the effect of operating

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frequency and source impedance on transfer function pole-zero location is investigated. It is shown that the low order models used in previous studies must be utilized with care since such models are not valid under all operating conditions.

Definitions and Basic Equations

In this paper, an idealized, symmetrical induction motor is assumed. This type of machine is generally defined to have the following characteristics: 1) uniform air gap; 2) linear magnetic circuit; 3) three identical stator windings distributed around the air gap so as to always produce a sinusoidal MMF wave in the air gap; 4) rotor bars or coils arranged so that the rotor MMF can also be considered a space sinusoid. The motor is assumed to be doubly-fed; supplied by balanced three-phase sets of voltages of synchronous and slip frequency with associated source impedance applied to the stator and rotor, respectively. The case of the singly-fed motor is obtained by simply setting the appropriate voltages and impedances to zero. Although the source voltages are balanced, it is assumed that each source is capable of instantaneous changes in amplitude, phase, or frequency. Sources with this characteristic are readily approximated by either an inverter or cycloconverter supply.

An equivalent circuit of this system referred to the synchronously rotating reference frame is given in Fig. 1. Associated with this figure, a time derivative operator of the form p/ω_b is assumed, where $p=d/dt$ and ω_b is base angular frequency. The figure also serves to define a number of the basic parameters and variables. Consistent with the above assumptions, the following matrix equations, expressed in SI units, can be derived which describe an idealized P-pole symmetrical induction machine in the synchronously rotating reference frame.^{4,5,17}

$$\bar{v}^e = \bar{R} \bar{i}^e + \frac{\omega}{\omega_b} \bar{F} \bar{i}^e + \frac{\omega_r}{\omega_b} \bar{G} \bar{i}^e - \frac{p}{\omega_b} \bar{X} \bar{i}^e \quad (1)$$

$$T_L = \frac{3}{2} \frac{(P/2)}{\omega_b} (\bar{i}^e)^T \bar{G} \bar{i}^e - \frac{\omega_b}{(P/2)} D \frac{\omega_r}{\omega_b} - \frac{\omega_b}{(P/2)} J \left(\frac{p}{\omega_b} \right) \frac{\omega_r}{\omega_b} \quad (2)$$

where

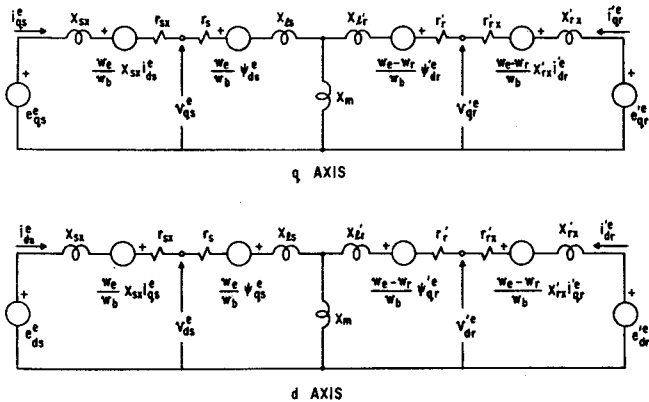


Fig. 1. Equivalent Circuit Synchronously Rotating Reference Frame

$$\bar{R} = \begin{bmatrix} r_s & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 \\ 0 & 0 & r_r' & 0 \\ 0 & 0 & 0 & r_r' \end{bmatrix} \quad \bar{X} = \begin{bmatrix} x_s & 0 & x_m & 0 \\ 0 & x_s & 0 & x_m \\ x_m & 0 & x_r' & 0 \\ 0 & x_m & 0 & x_r' \end{bmatrix}$$

$$\bar{F} = \begin{bmatrix} 0 & x_s & 0 & x_m \\ -x_s & 0 & -x_m & 0 \\ 0 & x_m & 0 & x_r' \\ -x_m & 0 & -x_r' & 0 \end{bmatrix} \quad \bar{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -x_m & 0 & -x_r' \\ x_m & 0 & x_r' & 0 \end{bmatrix}$$

$$\bar{v}^e = \begin{bmatrix} (\bar{v}_s^e)^T, (\bar{v}_r^e)^T \end{bmatrix}^T \quad \bar{i}^e = \begin{bmatrix} (\bar{i}_s^e)^T, (\bar{i}_r^e)^T \end{bmatrix}^T$$

and where

$$\bar{v}_s^e = \begin{bmatrix} v_{qs}^e \\ v_{ds}^e \end{bmatrix}^T \quad \bar{i}_s^e = \begin{bmatrix} i_{qs}^e \\ i_{ds}^e \end{bmatrix}^T$$

$$\bar{v}_r^e = \begin{bmatrix} v_{qr}^e \\ v_{dr}^e \end{bmatrix}^T \quad \bar{i}_r^e = \begin{bmatrix} i_{qr}^e \\ i_{dr}^e \end{bmatrix}^T$$

The prime indicates that this variable has been referred to the stator by the appropriate turns ratio. A T denotes the transpose. Also, a P-pole machine has been assumed. The actual mechanical angular speed has been referred to the speed of an equivalent two-pole machine ω_r . The stator line angular frequency is ω_e . Base frequency is ω_b . Stator self, rotor self, and mutual reactances x_s , x_r' , and x_m , respectively are expressed in terms of base frequency.

The voltages $(\omega_e/\omega_b) \bar{F} \bar{i}^e$ are "speed voltages" which arise as a result of the reference frame transformation and do not enter into the energy conversion process. The quantity $(\omega_r/\omega_b) \bar{G} \bar{i}^e$ corresponds to the speed voltages due to rotation and accounts for energy conversion. The terms in the torque equation, Eq. 2, can be identified as the load torque T_L , electromagnetic torque T_e , external damping torque T_d , and inertial torque T_i , respectively. The $3/2$ term which appears in the expression for electromagnetic torque is the result of using a transformation of variables which preserves the amplitude of both voltage and current when changing from phase to d-q quantities.

In addition to the motor equations, the effect of line impedance in the synchronously rotating reference frame is expressed

$$\bar{e}^e = \bar{R}_x \bar{i}^e + \frac{\omega}{\omega_b} \bar{F}_x \bar{i}^e + \frac{\omega_r}{\omega_b} \bar{G}_x \bar{i}^e + \frac{p}{\omega_b} \bar{X}_x \bar{i}^e + \bar{v}^e \quad (3)$$

where

$$\bar{R}_x = \begin{bmatrix} r_{sx} & 0 & 0 & 0 \\ 0 & r_{sx} & 0 & 0 \\ 0 & 0 & r'_{rx} & 0 \\ 0 & 0 & 0 & r'_{rx} \end{bmatrix} \quad \bar{X}_x = \begin{bmatrix} x_{sx} & 0 & 0 & 0 \\ 0 & x_{sx} & 0 & 0 \\ 0 & 0 & x'_{rx} & 0 \\ 0 & 0 & 0 & x'_{rx} \end{bmatrix}$$

$$\bar{F}_x = \begin{bmatrix} 0 & x_{sx} & 0 & 0 \\ -x_{sx} & 0 & 0 & 0 \\ 0 & 0 & 0 & x'_{rx} \\ 0 & 0 & -x'_{rx} & 0 \end{bmatrix} \quad \bar{G}_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x'_{rx} \\ 0 & 0 & x'_{rx} & 0 \end{bmatrix}$$

and

$$\bar{e}^e = \begin{bmatrix} (\bar{e}_s^e)^T, (\bar{e}_r^e)^T \end{bmatrix}^T; \quad \bar{e}_s^e = \begin{bmatrix} e_{qs}^e \\ e_{ds}^e \end{bmatrix}^T;$$

$$\bar{e}_r^e = \begin{bmatrix} e_{qr}^e \\ e_{dr}^e \end{bmatrix}^T$$

It is assumed that the motor is supplied by balanced sinusoidal voltages of the form $e_{as} = e_s \cos(\omega_e t + \alpha)$ for the stator and of the form $e_{ar} = e_r \cos[(\omega_e - \omega_r)t + \alpha + \sigma]$ for the rotor, where it is clear that the voltages e_{bs} , e_{cs} and e_{br} , e_{cr} are those required to form balanced polyphase sets.

In the synchronously rotating reference frame, the source voltages become

$$\bar{e}_s^e = \begin{bmatrix} e_s \cos \alpha \\ -e_s \sin \alpha \end{bmatrix}^T \quad (4)$$

$$\bar{e}_r^e = \begin{bmatrix} e_r \cos(\alpha + \sigma) \\ -e_r \sin(\alpha + \sigma) \end{bmatrix}^T$$

where the prime again accounts for the stator-rotor turns ratio transformation.

The angle α is an arbitrary displacement angle of the stator voltage space vector. It is conventional to assume that the qs-axis voltages is a positive maximum at $t=0$. That is, in the steady-state, α is chosen such that $\alpha_0=0$. The angle σ is the relative displacement of the rotor voltage space vector in the synchronously rotating reference frame relative to the stator source voltage vector.

Utilizing Eq. 3, the motor equations, Eq. 1, can be expressed in terms of the source voltages by

$$\bar{e}^e = (\bar{R} + \bar{R}_x) \bar{i}^e + \frac{\omega_e}{\omega_b} (\bar{F} + \bar{F}_x) \bar{i}^e + \frac{\omega_r}{\omega_b} (\bar{G} + \bar{G}_x) \bar{i}^e + \frac{p}{\omega_b} (\bar{X} + \bar{X}_x) \bar{i}^e \quad (5)$$

In partitioned form, Eq. 5 together with Eq. 2, is expressed

$$\begin{bmatrix} \bar{e}^e \\ T_l \end{bmatrix} = \begin{bmatrix} \bar{R}_t + \frac{\omega_e}{\omega_b} \bar{F}_t & \bar{G}_t \bar{i}^e \\ \frac{3}{2} \left(\frac{P/2}{\omega_b}\right) (\bar{i}^e)^T \bar{G} & -\frac{\omega_b}{(P/2)} D \end{bmatrix} \cdot \begin{bmatrix} \bar{i}^e \\ \frac{\omega_r}{\omega_b} \end{bmatrix} \quad (6)$$

$$+ \frac{p}{\omega_b} \begin{bmatrix} \bar{X}_t & \bar{0} \\ \bar{0}^T & -\frac{\omega_b^2}{(P/2)} J \end{bmatrix} \cdot \begin{bmatrix} \bar{i}^e \\ \frac{\omega_r}{\omega_b} \end{bmatrix}$$

where $\bar{R} + \bar{R}_x = \bar{R}_t$, $\bar{X} + \bar{X}_x = \bar{X}_t$, $\bar{G} + \bar{G}_x = \bar{G}_t$, $\bar{F} + \bar{F}_x = \bar{F}_t$ and $\bar{0}$ is a 4×1 column vector of zeros.

Equation 6 constitutes a set of nonlinear equations which completely describes transient behavior of the motor regardless of the terminal conditions. When the source voltages are balanced, the components of the voltage vector \bar{e}^e can be expressed in the form given by Eq. 4. When the load T_l and normalized line frequency ω_e/ω_b are also constant, the steady-state current vector and rotor speed become constant. Hence, the steady-state condition is given by the solution of the nonlinear algebraic equation

$$\begin{bmatrix} \bar{e}_o^e \\ T_{lo} \end{bmatrix} = \begin{bmatrix} \bar{R}_t + \frac{\omega_{eo}}{\omega_b} \bar{F}_t & \bar{G}_t \bar{i}_o^e \\ \frac{3}{2} \left(\frac{P/2}{\omega_b}\right) (\bar{i}_o^e)^T \bar{G} & -\frac{\omega_b}{(P/2)} D \end{bmatrix} \cdot \begin{bmatrix} \bar{i}_o^e \\ \frac{\omega_{ro}}{\omega_b} \end{bmatrix} \quad (7)$$

The subscript "o" is used to denote the steady-state value of the variable.

Perturbing all state and source variables of Eq. 6 by a small value from the steady-state solution, Eq. 7, the following equations describe behavior of the system for small changes around an operating point.

$$\begin{bmatrix} \Delta \bar{e}^e - \bar{F}_t \bar{i}_o^e \Delta \frac{\omega_e}{\omega_b} \\ \Delta T_l \end{bmatrix} = \begin{bmatrix} \bar{R}_t + \frac{\omega_{eo}}{\omega_b} \bar{F}_t + \frac{\omega_{ro}}{\omega_b} \bar{G}_t & \bar{G}_t \bar{i}_o^e \\ \frac{3}{2} \left(\frac{P/2}{\omega_b}\right) (\bar{i}_o^e)^T (\bar{G} + \bar{G}^T) & -\frac{\omega_b}{(P/2)} D \end{bmatrix} \cdot \begin{bmatrix} \Delta \bar{i}^e \\ \Delta \frac{\omega_r}{\omega_b} \end{bmatrix} \quad (8)$$

$$+ \frac{p}{\omega_b} \begin{bmatrix} \bar{X}_t & \bar{0} \\ \bar{0}^T & -\frac{\omega_b}{(P/2)} J \end{bmatrix} \cdot \begin{bmatrix} \Delta \bar{i}^e \\ \Delta \frac{\omega_r}{\omega_b} \end{bmatrix}$$

Equation 8 can be readily cast in standard state variable form by solving for the vector $p[(\Delta \bar{i}^e)^T, (\Delta \omega_r/\omega_b)]^T$, obtained by inverting the

4×4 reactance matrix. The resulting equation is

$$p \begin{bmatrix} \Delta \bar{i}^e \\ \Delta \frac{\omega_r}{\omega_b} \end{bmatrix} = \begin{bmatrix} -\omega_b \bar{X}_t^{-1} \left[\bar{R}_t + \frac{\omega_{eo}}{\omega_b} \bar{F}_t + \frac{\omega_{ro}}{\omega_b} \bar{G}_t \right] & -\omega_b \bar{X}_t^{-1} \bar{G}_t \bar{i}_o^e \\ \frac{3}{2} \left(\frac{P/2}{\omega_b}\right)^2 \frac{1}{J} (\bar{i}_o^e)^T (\bar{G} + \bar{G}^T) & -\frac{D}{J} \end{bmatrix} \begin{bmatrix} \Delta \bar{i}^e \\ \Delta \frac{\omega_e}{\omega_b} \\ \Delta T_l \end{bmatrix} \quad (9)$$

$$+ \begin{bmatrix} \Delta \bar{i}^e \\ \Delta \frac{\omega_r}{\omega_b} \end{bmatrix} + \begin{bmatrix} \omega_b \bar{X}_t^{-1} & -\omega_b \bar{X}_t^{-1} \bar{F}_t \bar{i}_o^e \\ \bar{0}^T & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta \bar{e}^e \\ \Delta \frac{\omega_e}{\omega_b} \\ \Delta T_l \end{bmatrix}$$

In compact form, Eq. 9 can be expressed

$$p \Delta \bar{x} = \bar{A} \Delta \bar{x} + \bar{B} \Delta \bar{u} \quad (10)$$

where the definition of $\Delta \bar{x}$, $\Delta \bar{u}$, \bar{A} and \bar{B} is implied by the context.

Transfer Function Input Δu

Equation 9 constitutes the linearized system differential equations for small arbitrary changes in all conventional motor inputs including stator voltage amplitude, stator phase angle or frequency, as well as rotor voltage amplitude, rotor phase angle and load torque. The frequency applied to the rotor, that is $\omega_e - \omega_r$ or slip angular frequency, has, however, been constrained to be a dependent variable since an independent change in both stator and rotor frequency implies that the stator and rotor MMFs are not synchronized. In some cases, slip frequency is considered the independent variable. In this case, the line frequency is constrained and the state equation must be altered. The proper state equation for independent changes in slip frequency is given in Appendix 1.

Equation 9 accounts for independent changes in all variables simultaneously. However, for the purpose of computing transfer functions, it is generally sufficient to consider arbitrary changes of each input individually. Equation 10 can be written in the form

$$p \Delta \bar{x} = \bar{A} \Delta \bar{x} + \bar{B} \bar{b} \Delta u \quad (11)$$

where Δu corresponds to a change in stator or rotor voltage amplitude or phase, stator frequency or load torque. Table 1 tabulates the corresponding vector \bar{b} for each input Δu . The subscript "o" again denotes the steady-state value of the variable.

Transfer Function Output Δy

Table 1 together with Eq. 9 serves to completely define the system equations for all conventional system inputs or, equivalently, all transfer function inputs. It is well known that any output of the system (hence, transfer function) can always be expressed as a linear combination of state variables and $\Delta \bar{u}$. That is to say

$$\Delta y = \bar{c}^T \Delta \bar{x} + \bar{d}^T \Delta \bar{u} \quad (12)$$

where in this equation $\Delta \bar{x}$ is again the state vector of currents together with rotor speed and $\Delta \bar{u}$ is the input vector of source voltage, line frequency, and load torque as defined by Eq. 9.

Output variables are, however, generally more directly related to terminal quantities of the machine than to the source quantities. It is desirable to define Δy as a linear combination of state variables $\Delta \bar{x}$ and terminal voltages $\Delta \bar{v}^e$ rather than $\Delta \bar{x}$ and $\Delta \bar{u}$. That is

$$\Delta y = \bar{c}_1^T \Delta \bar{x} + \bar{d}_1^T \Delta \bar{v}^e \quad (13)$$

Table 1 – System Inputs

Stator Inputs		Rotor Inputs	
Input Δu	\bar{b} vector	Input Δu	\bar{b} vector
Stator Voltage Amplitude Δe_s	$[1, 0, 0, 0, 0, 0]^T$	Rotor Voltage Amplitude Δe_r	$[0, 0, \cos \sigma_o, -\sin \sigma_o, 0, 0]^T$
Stator Voltage Phase $\Delta \alpha$	$[0, e_{s_o}, 0, 0, 0, 0]^T$	Rotor Voltage Phase $\Delta \sigma$	$[0, 0, -e_{r_o} \sin \sigma_o, -e_{r_o} \cos \sigma_o, 0, 0]^T$
Stator Frequency Δf_e	$[0, 0, 0, 0, \frac{1}{f_b}, 0]^T$	Load Torque ΔT_L	$[0, 0, 0, 0, 0, 1]^T$

This definition of Δy permits relatively simple expressions for the vectors \bar{c}_1 and \bar{d}_1 . Table 2 lists appropriate \bar{c}_1 and \bar{d}_1 vectors for most practical system (transfer function) outputs. Although the choice of useful feedback signals is limited only by the ingenuity of the control designer, the vast majority of control configurations employ feedback signals of terminal voltage, line current, input power, air gap flux, torque, or speed. It is clear that this table may be readily extended to include any output of interest.

In order to express the measurement equation in the standard form of Eq. 12, the terminal voltage vector $\Delta \bar{v}^e$ must be related to $\Delta \bar{x}$ and $\Delta \bar{u}$. When Eq. 1 is substituted into Eq. 3, the term $(p/\omega_b)\bar{i}^e$ eliminated and the resulting expression linearized, the motor terminal voltages can be expressed in terms of state and input variables as

$$\begin{aligned} \Delta \bar{v}^e = & \bar{x} \bar{x}_t^{-1} \left\{ \Delta \bar{e}^e + \left[(\bar{x}_t \bar{x}_t^{-1} \bar{R} - \bar{R}_t) + \frac{\omega}{\omega_b} (\bar{x}_t \bar{x}_t^{-1} \bar{F} - \bar{F}_t) \right. \right. \\ & + \frac{\omega e_o}{\omega_b} (\bar{x}_t \bar{x}_t^{-1} \bar{G} - \bar{G}_t) \left. \right] \Delta \bar{i}^e + (\bar{x}_t \bar{x}_t^{-1} \bar{F} - \bar{F}_t) \bar{i}_o^e \Delta \frac{\omega}{\omega_b} \\ & \left. + (\bar{x}_t \bar{x}_t^{-1} \bar{G} - \bar{G}_t) \bar{i}_o^e \Delta \frac{\omega}{\omega_b} \right\} \end{aligned} \quad (14)$$

Arranging Eq. 14 so that the state and input vectors appear explicitly and simplifying, Eq. 13 becomes

$$\begin{aligned} \Delta \bar{v}^e = & \left[\bar{R} + \frac{\omega e_o}{\omega_b} \bar{F} + \frac{\omega r_o}{\omega_b} \bar{G} - \bar{X} \bar{X}_t^{-1} (\bar{R}_t + \frac{\omega e_o}{\omega_b} \bar{F}_t + \frac{\omega r_o}{\omega_b} \bar{G}_t) \right. \\ & \left. (\bar{G} - \bar{X} \bar{X}_t^{-1} \bar{G}_t) \bar{i}_o^e \right] \Delta \left[(\Delta \bar{i}^e)^T, \Delta \frac{\omega}{\omega_b} \right]^T + \left[\bar{X} \bar{X}_t^{-1} (\bar{F} - \bar{X} \bar{X}_t^{-1} \bar{F}_t) \bar{i}_o^e, \bar{0} \right] \\ & \cdot \left[(\Delta \bar{e}^e)^T, \Delta \frac{\omega}{\omega_b}, \Delta T_L \right]^T \end{aligned} \quad (15)$$

It is clear that Eq. 15 is in the form

$$\Delta \bar{v}^e = \bar{C}_2 \Delta \bar{x} + \bar{D}_2 \Delta \bar{u} \quad (16)$$

where $\Delta \bar{x}$ and $\Delta \bar{u}$ are the state vector and input vector defined previously, and where, for a particular input $\Delta \bar{u} = \bar{b} \Delta u$ as given by Table 1. The definitions of \bar{C}_2 and \bar{D}_2 are clear from the context.

Table 2 – System Outputs

Output Δy	\bar{c}_1 vector	\bar{d}_1 vector	Remarks
Stator Voltage Amplitude Δv_s	$\bar{0}$	$[v_{qso}^e/v_{so}, v_{dso}^e/v_{so}, 0, 0]^T$	$v_{so} = \sqrt{(v_{qso}^e)^2 + (v_{dso}^e)^2}$
Rotor Voltage Amplitude $\Delta v_r'$	$\bar{0}$	$[0, 0, v_{qro}'/v_{ro}', v_{dro}'/v_{ro}']^T$	$v_{ro}' = \sqrt{(v_{qro}'^e)^2 + (v_{dro}'^e)^2}$
Stator Current Amplitude Δi_s	$[i_{qso}^e/i_{so}, i_{dso}^e/i_{so}, 0, 0, 0]^T$	$\bar{0}$	$i_{so} = \sqrt{(i_{qso}^e)^2 + (i_{dso}^e)^2}$
Rotor Current Amplitude $\Delta i_r'$	$[0, 0, i_{qro}'^e/i_{ro}', i_{dro}'^e/i_{ro}']^T$	$\bar{0}$	$i_{ro}' = \sqrt{(i_{qro}'^e)^2 + (i_{dro}'^e)^2}$
Real Component of Stator Current $\Delta i_{s,Re}$	$[\cos \varphi_{sx}, \sin \varphi_{sx}, 0, 0, 0]^T$	$\bar{0}$	$\varphi_{sx} = \tan^{-1}(v_{dso}^e/v_{qso}^e)$
Real Component of Rotor Current $\Delta i_{r,Re}'$	$[0, 0, \cos \varphi_{rx}, \sin \varphi_{rx}, 0]^T$	$\bar{0}$	$\varphi_{rx} = \tan^{-1}(v_{dro}'^e/v_{qro}'^e)$
Stator Power ΔP_s	$[\frac{3}{2} v_{qso}^e, \frac{3}{2} v_{dso}^e, 0, 0, 0]^T$	$[\frac{3}{2} i_{qso}^e, \frac{3}{2} i_{dso}^e, 0, 0]^T$	
Rotor Power $\Delta P_r'$	$[0, 0, \frac{3}{2} v_{qro}'^e, \frac{3}{2} v_{dro}'^e]^T$	$[0, 0, \frac{3}{2} i_{qro}'^e, \frac{3}{2} i_{dro}'^e]^T$	
Air Gap Flux Linkage $\Delta \lambda_m$	$\frac{X_m}{\omega_b i_{mo}} [i_{qso}^e + i_{qro}'^e, i_{dso}^e + i_{dro}'^e, i_{qso}^e + i_{qro}'^e, i_{dso}^e + i_{dro}'^e, 0]^T$	$\bar{0}$	$i_{mo} = \sqrt{(i_{qso}^e + i_{qro}'^e)^2 + (i_{dso}^e + i_{dro}'^e)^2}$
Electromagnetic Torque ΔT_e	$\frac{3P}{2} \frac{X_m}{\omega_b} [i_{dro}'^e, -i_{qro}'^e, -i_{dso}^e, i_{qso}^e, 0]^T$	$\bar{0}$	
Mechanical Speed $\Delta \omega_{rm}$	$[0, 0, 0, 0, \omega_b/(P/2)]^T$	$\bar{0}$	

Substituting Eq. 16 into 13, the output variable can be expressed in terms of $\Delta \bar{x}$ and $\Delta \bar{u}$ as

$$\Delta y = (\bar{c}_1^T + \bar{d}_1^T \bar{c}_2) \Delta \bar{x} + \bar{d}_1^T \bar{D}_2 \Delta \bar{u} \quad (17)$$

Hence, the output vectors \bar{c} and \bar{d} of Eq. 12 are defined as

$$\bar{c} = \bar{c}_1 + \bar{c}_2^T \bar{d}_1 \quad (18)$$

$$\bar{d} = \bar{D}_2^T \bar{d}_1 \quad (19)$$

Equations 10 and 17 together with Tables 1 and 2 form a basis by which the state equation and measurement equation can be derived for the generalized case of a doubly-fed induction motor with any systems input and output. Although these equations completely describe the system, they are unfortunately not in a form suitable for directly extracting the transfer function $\Delta y/\Delta u$. However, it can be shown that transformation exists by which these equations can be expressed in so-called normal or phase variable form. This form permits the coefficients of the characteristic polynomial and numerator polynomial to be obtained by inspection. The poles and zeros of the transfer function can then be readily obtained by any of a number of conventional polynomial factorization algorithms. In particular, the downhill-Newton method has been used by the authors. A discussion of the significance of the transformation to phase variable form together with a description of the transformation algorithm is given in the Appendix 2.

Comparison with Simulation Results

In the preceding analysis, the method of small displacements together with state variable techniques, have been used to derive a generalized method for obtaining induction motor transfer functions. It is of interest to compare the results of this method which requires linearization of the motor equations to a detailed solution of the actual nonlinear motor equations. The machine selected for purposes of comparison was a conventional, singly-fed, 110 hp, 210 V/phase, 4 pole, 50 Hz, squirrel-cage induction machine. The measured motor parameters expressed in SI units are $r_s=0.021$, $x_s=4.207$, $r_r=0.017$, $x_r=4.316$, $x_m=4.14$. The inertia of the motor plus connected load was taken as 5 kg m^2 which corresponds to an inertia constant $H=0.8$. The effect of external damping torque as well as the effects of source impedance was neglected in this example. Thus Δv_s is equal to Δe_s .

Table 3 shows transfer functions computed for the operating point $T_{e0}=1000 \text{ N}\cdot\text{m}$, $f_{e0}=50 \text{ Hz}$, $e_{s0}=v_{s0}=296.9 \text{ V pk}$. The corresponding stator current and slip frequency for this case is $i_{s0}^s=365$, $i_{d0}^s=191$, $i_{q0}^s=412 \text{ A pk}$, and $f_{s0}=f_{e0}-f_{ro}=1.25 \text{ Hz}$. Transfer function $\Delta T_e/\Delta v_s$ is shown both for infinite inertia and the rated inertia. The corresponding time domain solutions for a 10 V step change in v_s is also given.

Comparing the two corresponding solutions, a marked difference in the transfer function can be noted. For the case of constant speed, the transfer function $\Delta T_e/\Delta v_s$ has 4 poles and 3 zeros in the left half-plane. The gain constant indicates that for increases in voltage, torque will increase in the steady state. When speed is permitted to change, the transfer function is characterized by 5 poles and 4 zeros. One zero exists in the right half-plane resulting in a non-minimum phase transfer function. Also, the gain constant has changed sign indicating that the torque, while initially increasing, will ultimately decrease for a step increase in voltage amplitude.

This transfer function behavior can be justified from practical considerations. When speed is held constant and the terminal voltage is increased, the current increases as expected for a linear, passive system. The increase in current causes a corresponding increase in torque. When the constant speed constraint is removed, this increase in torque results in a speed increase. The resulting change in electromagnetic torque after the transient has occurred is ideally zero since the load torque is assumed fixed. The stator current eventually decreases since the motor is now operating under a higher flux condition. It should be noted that the steady-state gain of $\Delta T_e/\Delta v_s$ should ideally be zero indicating zero net change in torque after a voltage disturbance. The small negative value in the actual transfer function can be attributed to the effect of linearizing the electromagnetic torque equation.

In Fig. 2, the time response of torque to a 10 volt step change in voltage amplitude has been plotted. Responses for the case of both infinite and finite inertia ($J=5 \text{ kg m}^2$) are shown. The solid line shows the results of a detailed digital computer simulation which preserves the nonlinear terms in the motor equations.¹⁷ The points were calculated from the explicit time response given in Table 3. Very acceptable correlation is evident. Similar correlation has been obtained for other transfer functions.

Comparison of Reduced Order Transfer Functions

It has long been an accepted practice to assume constant speed for purposes of a control study. Comparing the transfer functions for the two cases of infinite and finite inertia, however, it appears that the character of the transfer functions for the two cases is substantially

Table 3 — Transfer Functions and Time Response for Step Changes in Voltage Amplitude, $\Delta v_s=10 \text{ V}$. $T_{e0}=1000 \text{ N}\cdot\text{m}$, $f_{e0}=50 \text{ Hz}$, $v_{s0}=296.9 \text{ V}$.

	Transfer Function	Time Response for $\Delta v_s (s)=10/s$
Constant Speed $J = \infty$	$\frac{\Delta T_e}{\Delta v_s} = \frac{6.74(1+s/123)(1+0.96s/31.8+s^2/31.8^2)}{(1+1.83s/24+s^2/24^2)(1+0.178s/314+s^2/314^2)}$	$\Delta T_e = 67.4+102.7 \exp(-22.0t) \sin(9.67t-2.81) + 102.9 \exp(-28.0t) \sin(312t-0.34)$
Finite Inertia $J=5 \text{ kg m}^2$	$\frac{\Delta T_e}{\Delta v_s} = \frac{-0.082(1-s/0.46)(1+s/123)(1+0.96s/32.0+s^2/32.0^2)}{(1+s/17.7)(1+0.736s/35.3+s^2/35.3^2)(1+0.18s/314+s^2/314^2)}$	$\Delta T_e = -0.82+11.0 \exp(-13.0t) \sin(32.8t+1.5) + 24.7 \exp(-17.7t)+104 \exp(-28.2t) \sin(312t-0.34)$
Finite Inertia Speed Pole and Zero Cancelled	$\frac{\Delta T_e}{\Delta v_s} = \frac{3.19(1+s/123)(1+0.96s/32.0+s^2/32.0^2)}{(1+0.736s/35.3+s^2/35.3^2)(1+0.18s/314+s^2/314^2)}$	$\Delta T_e = 31.9+10.3 \exp(-13.0t) \sin(32.8t+0.97) + 104 \exp(-28.2t) \sin(312t-0.4)$
Finite Inertia Rotor Circuit Pole and Zero Cancelled	$\frac{\Delta T_e}{\Delta v_s} = \frac{3.89(1+s/123)}{(1+0.18s/314+s^2/314^2)}$	$\Delta T_e = 38.9+103 \exp(-28.2t) \sin(312t-0.38)$

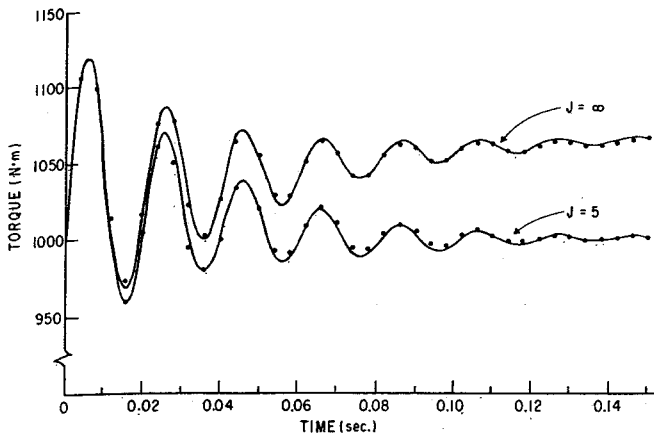


Fig. 2. Torque Response for a Step Change in Voltage Solid Line-Exact Solution; Points-Using Transfer Function

different, particularly the steady-state gain (SSG). A root locus study shows, however, that either transfer function produces substantially the same root contours.

Correlation between the two cases is more readily apparent if the pole and zero resulting from speed variations is cancelled from the more detailed transfer function. The resulting transfer function and corresponding time response is shown in the third row of Table 3. It can be observed that the remaining poles, zeros and gain are now similar to the constant speed case. The most significant difference is the steady-state gain and location of the rotor poles. Since the rotor zeros are located near the rotor poles, a further simplification in the transfer function can be made by cancelling the rotor poles and zeros. Transfer functions which account only for stator dynamics is given in the fourth row of Table 3.

Simplified formulations permit the implementation of a control design by simple graphical methods. However, it is important to bear in

mind that an assumption of constant speed neglects possible instability mechanisms. Inaccuracy in the SSG (3.19 vs. 6.74) combined with the error in the estimate of the damping of the rotor poles ($\zeta=0.37$, $\omega_n=35.3$ vs. $\zeta=0.91$, $\omega_n=24$) can result in a poorly damped or even unstable system if the control system crossover frequency is chosen near 35 rad/s. Also, an assumption of constant speed results in a transfer function which is of minimum phase with an excess of poles to zeros of one. Hence, it appears that the gain of the feedback system between torque and voltage can be increased indefinitely without danger of instability. However, the transfer function is actually non-minimum phase. As the feedback gain increases, the closed loop poles approach the location of the open loop zeros. The rotor speed induced pole moves toward the non-minimum phase zero and will cross over into the right half-plane. It will be subsequently shown that the value of gain needed to produce instability is not necessarily excessive.

Comparison of Transfer Function Zeros

Thus far, only one transfer has been investigated in detail. However, the state variable method provides a means by which nearly any transfer function can be evaluated with ease. In Tables 4 and 5 are compiled the steady-state gain (SSG) and zeros for a variety of transfer functions. In order to restrict the possible combinations to a reasonable degree, the same singly-fed induction motor used previously was chosen. However, in this case a finite source impedance was assumed wherein $r_{sx}=0.02$, $x_{sx}=0.125$. The source voltage e_{s0} was adjusted such that the motor terminal voltage was maintained at its rated value ($v_{s0}=296.9$ V pk). Again $T_{e0}=1000$ N·m, $f_{e0}=50$ Hz.

Using the approach of this paper, all possible combinations of four inputs Δe_s , Δf_e , $\Delta \alpha$ and ΔT_e and 7 outputs Δv_s , Δi_s , $\Delta i_{s,Re}$, ΔP_s , $\Delta \lambda_m$, ΔT_e and $\Delta \omega_{rm}$ were computed for both finite and infinite inertia. The digital computer processing time required to compute the 52 possible transfer functions was less than 20 seconds using a GE 600 series computer. In some cases, attention must be paid to round off.

Table 4 - Gain and Zeros for Infinite Inertia; $J=\infty$. Operating Point: $v_{s0}=296.9$ V; $T_{e0}=1000$ N·m; $f_{e0}=50$ Hz. Source Impedance: $r_{sx}=0.02 \Omega$, $x_{sx}=0.125 \Omega$. Poles for all Transfer Functions: $-14.7 \pm j93.5$; $-35.8 \pm j312.6$.

Output	Input Δe_s		Input Δf_e	
	SSG	Zeros	SSG	Zeros
Δv_s	0.90	$-18.4 \pm j9.37$; $-31.9 \pm j311.9$	-26	$-30.8 \pm j341.5$; -38.9
Δi_s	1.25	$-6.63 \pm j12.7$; -258	237	415; -16.7
$\Delta i_{s,Re}$	1.10	$-2.04 \pm j20.9$; -87.7	176	478; -12.2 ; -757
ΔP_s	983	$-10.8 \pm j19.6$; -167 ; -1074	6978	507; -10.7 ; -809
$\Delta \lambda_m$	$2.7 \cdot 10^{-3}$	1511; $-23.0 \pm j6.66$	-0.143	-46.1 ; -98.6
ΔT_e	6.05	$-11.6 \pm j20.1$; -176	385	591; -10.1

Table 5 - Gain and Zeros of Motor Transfer Functions for Finite Inertia; $J=5.0$ kg m². Operating Point: $v_{s0}=296.9$ V; $T_{e0}=1000$ N·m, $f_{e0}=50$ Hz. Source Impedance: $r_{sx}=0.02$, $x_{sx}=0.125 \Omega$. Poles for all Transfer Functions: -9.38 , $-9.93 \pm j27.9$; $-35.9 \pm j312.6$.

Output	Input Δe_s		Input Δf_e	
	SSG	Zeros	SSG	Zeros
Δv_s	1.27	-17.7 ; $-9.81 \pm j28.6$; $-31.6 \pm j312.3$	-2.26	-2.9 ; -30.7 ; $-33.5 \pm j344.2$
Δi_s	-2.27	12.9; -259 ; $-12.6 \pm j24.9$	14.3	414; -1.90 ; -14.3
$\Delta i_{s,Re}$	-1.52	22.8; -79.6 ; $-17.5 \pm j22.0$	9.47	478; -1.55 ; -11.4 ; -756
ΔP_s	-62.0	1.46; $-12.2 \pm j20.8$; -166 ; -1073	3502	507; -1.38 ; -10.5 ; -808
$\Delta \lambda_m$	$4.5 \cdot 10^{-3}$	1511; $10.0 \pm j29.5$; -26.6	-0.028	-5.17 ; -56.2 ; -83.5
ΔT_e	-0.018	0.082; $-11.6 \pm j20.1$; -175.7	-0.07	592; 0.005; -10.1
$\Delta \omega_{rm}$	$9.0 \cdot 10^{-4}$	$-11.6 \pm j20.1$; -176	2.86	592; -101

Table 6 – Poles and Zeros of Transfer Function $\Delta i_s/\Delta e_s$ with Changes in Source Impedance for Constant Terminal Voltage. Operating Point: $v_{s0}=296.9V$, $T_{e0}=1000 N\cdot m$, $f_{e0}=50$ Hz. Inertia: $J=5$ kg m^2 .

	SSG	Zeros	Poles
$r_{sx}=0; x_{sx}=0$	-1.81	14.1; -14.9+j31; -200	-13.0+j32.8; -17.7; -28.2+j312.3
$r_{sx}=0.02; x_{sx}=0.125$	-2.27	12.9; -12.6+j24.9; -259	-9.93+j27.9; -9.37; -35.9+j312.5
$r_{sx}=0.04; x_{sx}=0.25$	-3.27	12.2; -11.2+j21.8; -312	-8.65+j25.0; -4.95; -39.7+j312.8
$r_{sx}=0.06; x_{sx}=0.375$	-6.32	11.9; -10.4+j20.2; -362	-8.02+j23.1; -2.12; -41.8+j313.1
$r_{sx}=0.08; x_{sx}=0.5$	-89.2	11.6; -9.75+j19.1; -409	-7.67+j21.8; -0.127; -43.3+j313.2

Table 7 – Poles and Zeros of Transfer Function $\Delta T_e/\Delta f_{s\ell}$ at Various Operating Frequencies. Assumptions: No Source Impedance; Rated Air Gap Flux; $T_{e0}=1050$ N·m; $J=5$ kg m^2 .

f_{e0}	SSG	Zeros	Poles
100	-88	1775; 0.18; -19.2	-1.4; -21.7+j7.63; -27.6+j628
50	2.92	880; -0.01; -18.6	-3.0; -21.2+j7.28; -27.3+j314
25	-0.016	432; $1.5 \cdot 10^{-4}$; -17.7	-7.3; -19.6+j5.13; -26.8+j157
17.5	-0.24	297; $3.7 \cdot 10^{-3}$; -17.7	-23; -12.0+j6.39; -26.4+j109
10	0.026	159; $-9.5 \cdot 10^{-4}$; -16.9	-30; -8.9+j12.7; -26.0+j61.8
5	-0.004	56; $2.3 \cdot 10^{-3}$; -14.3	-40; -2.7+j15.4; -27.3+j33.5
2.5	$3.0 \cdot 10^{-5}$	$0.65+j141$; $2.7 \cdot 10^{-6}$	-47; -0.54+j12.6; -27.3+j27.3

It was assumed that zeros whose magnitude is greater than 10^6 are at infinity.

Transfer functions for two of the four inputs are shown in Tables 4 and 5. A vast difference in the zero locations for different transfer functions is apparent. It can be noted that many transfer functions change substantially when the constant speed constraint is removed (e.g., $\Delta i_s/\Delta e_s$, $\Delta T_e/\Delta f_e$) whereas other transfer functions are not materially affected (e.g., $\Delta P_s/\Delta f_e$, $\Delta \lambda_m/\Delta f_e$). Several transfer functions display non-minimum phase characteristics for the finite inertia case only (e.g., $\Delta i_s/\Delta e_s$, $\Delta P_s/\Delta e_s$) whereas others are non-minimum phase for both cases (e.g., $\Delta \lambda_m/\Delta e_s$, $\Delta P_s/\Delta f_e$). Still other transfer functions remain minimum phase with or without the constant speed constraint (e.g., $\Delta v_s/\Delta e_s$, $\Delta \lambda_m/\Delta f_e$).

In the design of feedback compensators, it is conventional to neglect remote poles or zeros or utilize pole-zero cancellation techniques so as to simplify compensator design. From Tables 4 and 5 it is evident that no single approximation is valid for all motor transfer functions. However, the state variable method provides a convenient means by which an exact transfer function can be derived. Simplifications can then be made for each operating point with a precise knowledge of the severity of the approximation and its consequent effect on the resulting analysis.

Effect of Source Impedance

Because of the difficulty involved, source impedance is typically neglected in a control analysis. In Table 6, the poles and zeros for the transfer function $\Delta i_s/\Delta e_s$ have been tabulated for source impedance ranging from zero to approximately twice the motor leakage reactance. The source voltage e_{s0} has been adjusted so as to maintain the terminal voltage v_{s0} constant at 296.9 V. It is apparent that an increase in source impedance has a major effect on the location of the pole produced by speed changes. It is clear that increasing the source impedance severely limits the feedback gain that can be employed for this transfer function. Neglecting either the effect of impedance or speed changes can easily result in the design of an unstable system.

Effect of Line Frequency

Changes in operating frequency can also be shown to have an important effect on the behavior of motor transfer functions. Recently, much effort has been directed to the area of adjustable frequency,

induction motor speed control^{6-8,10,11,13,14} In order to effect an analysis, various assumptions have been made by these authors. In two of the above papers^{8,13}, it has been suggested that the transfer function $\Delta T_e/\Delta f_{s\ell}$ can be adequately represented by a transfer function of the form

$$\frac{\Delta T_e}{\Delta f_{s\ell}} = \frac{(\partial T_e / \partial f_{s\ell})_{n.l.}}{1 + T_r^1 s} \quad (20)$$

where $f_{s\ell} = f_e - f_r$ is the rotor slip frequency, $(\partial T_e / \partial f_{s\ell})_{n.l.}$ is the slope of the steady-state, no load characteristic of T_e vs. $f_{s\ell}$ and T_r^1 is the rotor transient time constant.

It is of interest to compare this simplified transfer function with the corresponding detailed representation. It can be readily established that for the motor parameters used in this paper, the proper simplified expression is given by

$$\frac{\Delta T_e}{\Delta f_{s\ell}} = \frac{959}{1+s/22.4} \quad (21)$$

Table 7 shows a compilation of the poles and zeros for this same transfer function using the state variable approach. A range of operating frequencies have been selected to correspond to a typical speed range. In order to maintain the same operating condition as frequency changes, the terminal voltage has been adjusted so as to maintain constant air gap flux.

Comparing the transfer function at 100 Hz to the simplified expression, it can be observed that the suggested approximation involves neglecting the zero at 1775 as well as the complex conjugate stator circuit poles located at -27.6+j628. The zero at 0.18 has been cancelled with the speed induced pole at -1.4, and finally the zero at -19.1 has been cancelled with one of the rotor circuit poles located at -21.7+j7.63.

It can be seen that these approximations remain valid with changes in line frequency to approximately 18 Hz. Below this point, the damping corresponding to the rotor poles is substantially reduced indicating that the transfer function is no longer adequately represented by a first order system. The speed induced pole moves to the left whereas the corresponding zero remains fixed, indicating that cancellation of this pair becomes invalid. The inadequacy of this simplified expression for $\Delta T_e/\Delta f_{s\ell}$ has also previously been identified qualitatively¹³.

Other system parameters including rotor resistance and load torque can also be shown to have important effects on transfer function behavior. It is clear that caution should be exercised in the application of simplified models since they can be easily used in situations where the simplifying assumptions do not apply.

CONCLUSION

This paper has presented a new, state variable approach to the derivation of induction motor transfer functions. Although the paper has been limited to induction motors, it is clear that the principle can readily be extended to reluctance-synchronous or synchronous machines. The method described herein has eliminated the need for manual derivation of transfer functions and with it the need for simplifying assumptions in their derivation. Since the entire derivation procedure can be relegated to a digital computer; complete, detailed motor transfer functions can be obtained without difficulty. Simplifications, if desired, can then be made with the analyst alerted to its possible effect on the subsequent analysis. It is expected that the state variable approach to motor transfer functions will find widespread application in the field of ac motor control.

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APPENDIX I

When slip frequency rather than stator frequency is considered the independent variable, the state equation is

$$p \begin{bmatrix} \Delta \bar{i}^e \\ \frac{\Delta \omega_r}{\omega_b} \end{bmatrix} = \begin{bmatrix} -\omega_b \bar{x}_t^{-1} \left[\bar{R}_t + \frac{\omega_{e0}}{\omega_b} \bar{F}_t + \frac{\omega_{r0}}{\omega_b} \bar{G}_t \right] & -\omega_b \bar{x}_t^{-1} (\bar{F}_t + \bar{G}_t) \bar{i}_o^e \\ \frac{3}{2} \left(\frac{P}{2} \right)^2 \frac{1}{J} (\bar{i}_o^e)^T (\bar{G} + \bar{G}^T) & -D/J \end{bmatrix} \begin{bmatrix} \Delta \bar{i}^e \\ \frac{\Delta \omega_{s\ell}}{\omega_b} \end{bmatrix} + \begin{bmatrix} \omega_b \bar{x}_t^{-1} & -\omega_b \bar{x}_t^{-1} \bar{F}_t \bar{i}_o^e & \bar{0} \\ \bar{0}^T & 0 & -P/2 \\ \bar{0}^T & 0 & \omega_b J \end{bmatrix} \begin{bmatrix} \Delta \bar{e}^e \\ \frac{\Delta \omega_{s\ell}}{\omega_b} \\ -\frac{\Delta T}{\Delta T} \ell \end{bmatrix} \quad (22)$$

where

$$\frac{\Delta \omega_{s\ell}}{\omega_b} = \frac{\Delta(\omega_e - \omega_r)}{\omega_b} \quad (23)$$

It can be noted that Eq. 23 constitutes a constraint on a state variable which alters the poles rather than the zeros of the transfer function. The analysis of this paper applies equally for the case of slip frequency input if the matrix A is defined by Eq. 22 and the vector $\bar{b} = [0, 0, 0, 0, 1/\omega_b, 0]^T$ as defined by Table 1 corresponds to an input of slip frequency rather than line frequency.

APPENDIX II

It is well known that any set of time invariant linear differential equations having a single forcing function can be expressed in the form

$$p\bar{x} = \bar{A}\bar{x} + \bar{b}u \quad (24)$$

where \bar{A} is an nxn matrix, \bar{b} an nxl vector of constant coefficients, \bar{x} an nxl vector of state variables, and u is the input to the system. Equation 24 is termed the state equation of the system. Since the vector \bar{x} completely describes the state of the system for any time $t \geq 0$, any physical variable, not identically one of the state variables can always be expressed as an algebraic combination of the state variables \bar{x} and input u . That is to say,

$$y = \bar{c}^T \bar{x} + d u \quad (25)$$

where \bar{c} is an nxl vector of constant coefficients and d is a scalar. Equation 25 is termed the measurement equation.

It is assumed that it is desired to find the transfer function y/u . It is clear that Eqs. 24 and 25 also represent the corresponding Laplace transformed equations if s is substituted for p and zero initial conditions are assumed. The difficulty involved in determining the transfer function y/u can be demonstrated by formally attempting to find a solution. Solving for \bar{x} in Eq. 24

$$(s\bar{I} - \bar{A})\bar{x} = \bar{b}u \quad (26)$$

where \bar{I} is the nxn identity matrix. Equation 26 is simply a set of linear

equations with n unknowns. Applying Cramer's rule, the solution for any specific x_j in terms of the input u can be obtained as

$$\frac{x_j(s)}{u} = \frac{|s\bar{I}-\bar{A}|_j}{|s\bar{I}-\bar{A}|} \quad j = 1, 2, \dots, n \quad (27)$$

The subscript j denotes that column j of the determinant $|s\bar{I}-\bar{A}|$ has been replaced by column vector \bar{b} .

The output vector can now be expressed

$$\frac{y(s)}{u(s)} = \frac{c_1 |s\bar{I}-\bar{A}|_1 + c_2 |s\bar{I}-\bar{A}|_2 + \dots + c_n |s\bar{I}-\bar{A}|_n + d |s\bar{I}-\bar{A}|}{|s\bar{I}-\bar{A}|} \quad (28)$$

The denominator of (27) implies an n th order polynomial in s . A solution of the denominator is not difficult since eigenvalue subroutines are readily available to solve this type of equation. However, the j th numerator term in Eq. 28 is not of proper form since the j th column of $|s\bar{I}-\bar{A}|$ has been replaced by the vector \bar{b} . The numerator determinants can be manipulated so as to extract the coefficients for each polynomial. However, since the manipulation must be repeated n times, the approach is tedious even with a digital computer.

The complexity of extracting the transfer function is simplified if another approach is taken. From the form of Eq. 28, it is clear that the transfer function y/u results in an equation of the form

$$\frac{y(s)}{u(s)} = \frac{n_1 + n_2 s + \dots + n_n s^{n-1}}{m_1 + m_2 s + \dots + m_n s^{n-1} + s^n} + d \quad (29)$$

Define an auxiliary variable x_{1p} such that

$$\frac{x_{1p}(s)}{u(s)} = \frac{1}{m_1 + m_2 s + m_3 s^2 + \dots + m_n s^{n-1} + s^n} \quad (30)$$

Then

$$u(s) = (m_1 + m_2 s + \dots + m_n s^{n-1} + s^n) x_{1p}(s) \quad (31)$$

$$y(s) = (n_1 + n_2 s + \dots + n_n s^{n-1}) x_{1p}(s) + du(s) \quad (32)$$

Also, define $n-1$ additional variables such that

$$x_{2p} = s x_{1p}; x_{3p} = s x_{2p}; \dots; x_{np} = s x_{(n-1)p} \quad (33)$$

Equations 31-33 lead to a state and measurement equation of the form

$$s \begin{bmatrix} x_{1p} \\ x_{2p} \\ \vdots \\ x_{np} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -m_1 & -m_2 & \dots & \dots & -m_n \end{bmatrix} \begin{bmatrix} x_{1p} \\ x_{2p} \\ \vdots \\ x_{np} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \quad (34)$$

$$y = [n_1, n_2, \dots, n_n] \begin{bmatrix} x_{1p} \\ x_{2p} \\ \vdots \\ x_{np} \end{bmatrix} + du \quad (35)$$

or simply

$$s \bar{x}_p = \bar{A}_p \bar{x}_p + \bar{b}_p u \quad (36)$$

$$y = \bar{c}_p^T \bar{x}_p + du \quad (37)$$

Eqs. 34 and 35 are referred to as the phase variable or normal form of the state equations. It is clear that if the original equations are manipulated into the above form, the coefficients of the transfer function numerator and denominator can be obtained by inspection.

In order to accomplish the necessary change of variable, it is necessary to find a non-singular linear transformation of the form

$$\bar{x} = \bar{P} \bar{x}_p \quad (38)$$

The coefficient matrices for the phase variable form of the state and measurement equations are related to the untransformed equations by

$$\bar{A}_p = \bar{P}^{-1} \bar{A} \bar{P}; \quad \bar{b}_p = \bar{P}^{-1} \bar{b}; \quad \bar{c}_p^T = \bar{P}^T \bar{c} \quad (39)$$

It has been shown by Tuel¹⁸ and Rane¹⁹ that if the coefficients of the characteristic polynomial of \bar{A} , m_k , $k=1, \dots, n$ are known, then the transformation matrix \bar{P} can be obtained from a recursion relationship. The coefficients m_k , as defined by Eq. 29, are readily obtained by the method of principal minors or Leverrier's method. A FORTRAN computer program employing the method of principal minors together with the algorithm of Tuel and Rane have been written by Melsa²⁰.

The algorithm is summarized as follows. Define \bar{P} as a series of column vectors

$$\bar{P} = [\bar{p}_1, \bar{p}_2, \bar{p}_3, \dots, \bar{p}_n] \quad (40)$$

The columns of \bar{P} are found from the following recursion relationship

$$\bar{p}_n = \bar{b} \quad (41)$$

$$\bar{p}_{n-k} = \bar{A} \bar{p}_{n-k+1} + m_{n-k+1} \bar{b}; \quad k = 1, 2, \dots, n-1 \quad (42)$$