A MODAL APPROACH TO THE TRANSIENT ANALYSIS OF SYNCHRONOUS MACHINES

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SUMMARY

A novel method for calculating the transient currents, flux linkages and electromagnetic torque of a synchronous machine following a sudden voltage disturbance is presented. The solution is obtained for a sudden change from an arbitrary balanced to an arbitrary unbalanced voltage condition. Since minimum constraints are placed on the disturbance, many typical types of transients such as three-phase and line-to-line short-circuit faults can be accommodated. Complete solutions for all machine currents are obtained as explicit functions of time. Any number of rotor circuits can be included and full allowance is inherently made for coupling between all stator and rotor circuits. Constant speed is assumed but no additional approximations are made other than those implicit in the derivation of Park's Equations. The technique incorporates the state space concept of modal analysis in its implementation. Closed form solutions obtained by the modal theory are compared with standard solution techniques.

INTRODUCTION

A short-circuit fault at the terminals of a synchronous machine is considered as a severe test of machine performance and is commonly used as a criterion for characterizing transient behavior of a synchronous machine. In general, the equations which describe the performance of a synchronous machine are nonlinear and, therefore, can only be solved by analog or digital computer simulation techniques. Such a solution is particularly useful when the behavior of the machine is evaluated over an extended time interval. However, transient currents are typically required only for a few cycles immediately after the fault, so that a closed form solution becomes feasible by assuming constant speed during the period of interest. Over the years closed form solutions have been obtained for a number of typical occurrences such as three-phase faults, line-to-ground faults, etc. These solutions are obtained by solving Park's Equations algebraically and are well documented in the literature[1,2]. A particular advantage of closed-form solutions is that they provide insight into the performance of the machine not possible with simulation.

To carry out an explicit solution, the circuit model must be simplified so that the algebra is kept within a manageable scale. For example, only one amortisseur circuit is generally included on each axis and the effect of stator resistance is often neglected. Approximations are also made in manipulating the algebra even when only one damper circuit is considered. For example, the so-called "transient reactances" x_d', x_d'' are defined by assuming that the rotor leakage reactances are much larger than the rotor resistances. Solutions are usually

Electric Machines and Electromechanics: An International Quarterly, 1:337-354, 1977

given only for the case of an unloaded machine (δ=0) with zero prefault current which again reduces the manipulation involved.

These approximations are generally valid for conventional machines equipped with definite amortisseur windings. However, some machines, for example solid rotor turbogenerators, necessitate more detailed rotor models having multiple rotor circuits[3]. The numerous time constants associated with each rotor circuit together with the extensive coupling among circuits make the usual simplifying approximations difficult to justify. Analytical expressions for conventional quantities such as x_d', x_d'', τ_d' and τ_d'' become too complicated for practical use.

In addition to machines which require detailed models, new applications are arising which make simplified solutions invalid even for conventional synchronous machines. One of these applications which has received much attention recently is the variable-speed load-commutated synchronous motor drive. Such drives must typically operate under load to less than 10% of rated speed. At these low speeds the resistances of the machine can equal or even exceed the corresponding leakage reactances. Since the usual assumptions do not apply, solutions found in the literature become less accurate. The complexity of the algebra involved in obtaining the correct solution increases greatly to the point where the feasibility of an explicit form solution must be questioned.

This paper presents an alternative approach for developing closed form solutions to synchronous machine transients. The method is based on modern eigenvalue-eigenvector techniques often referred to as modal theory[4]. Although the modal theory has been used in the past for machine analysis, application has been mainly directed to the study of transients resulting from small displacements from a steady-state operating condition (i.e., dynamic stability)[5,6]. The notable exception appears to be the paper by Charlton[7] which develops the specific solution for the case of a symmetric three-phase fault for a round-rotor machine with identical d- and q-axis rotor parameters. In this paper the general solution is obtained for the case of a sudden change from a balanced, loaded, steady-state operating condition to an arbitrary unbalanced condition. Hence, the correct result for most types of fault conditions can be obtained simply by specifying the proper winding voltages after the fault occurs. Moreover, solutions for other important transients can also be obtained without difficulty. Among these include voltage dips and/or phase shifts resulting from breaker operation or changes in load, step changes in field excitation which occur in conjunction with voltage dips, synchronizing out of phase, and both transient and steady-state currents resulting from sudden phase unbalances. The transient solution for all winding currents is obtained directly as an explicit function of time consisting of sinusoidal and exponential terms with rates of decay corresponding to the armature, transient and subtransient time constants. The solution is written in a form such that the proper weighting factors for each transient component can be readily computed.

d-q REPRESENTATION OF SOURCE VOLTAGES

A simplified diagram of the system to be analyzed is given in Fig. 1. In general, three sinusoidal voltages of arbitrary phase and amplitude are assumed connected to the terminals of a three phase, four wire synchronous machine. Before the disturbance the machine is assumed to be operating synchronously with the applied voltages. The quantities r_g and x_g are the parameters of the neutral grounding reactor connected between the machine neutral "s" and the ground point of the bus voltages "g". For simplicity no impedance is assumed external to the machine but, if desired, a balanced set of external impedances can be lumped with the stator impedance of the machine.

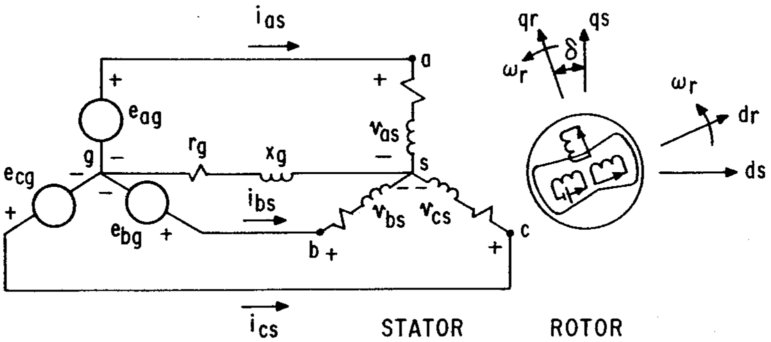

Fig. 1. Synchronous machine with arbitrary set of sinusoidal voltages.

The three source voltages both before and after the transient are, in general, described by

$$e_{ag} = E_{a\alpha}\cos\omega_e t + E_{a\gamma}\sin\omega_e t \quad (1)$$

$$e_{bg} = E_{b\alpha}\cos\omega_e t + E_{b\gamma}\sin\omega_e t \quad (2)$$

$$e_{cg} = E_{c\alpha}\cos\omega_e t + E_{c\gamma}\sin\omega_e t \quad (3)$$

The subscripts α and γ denote the cosine and sine terms of the source voltages respectively[8,9]. The three stator phase voltages are defined by

$$v_{as} = e_{ag} - v_{sg} \quad (4)$$

$$v_{bs} = e_{bg} - v_{sg} \quad (5)$$

$$v_{cs} = e_{cg} - v_{sg} \quad (6)$$

The equations relating the phase voltages to d-q voltages expressed in a reference frame fixed on the stator can be found from Ref. 10, Eqs. 28-30 by aligning the q-axis of the arbitrary reference frame along the phase a axis (θ=0) and setting the speed of the reference frame to zero (pθ=0) whereupon

$$v_q^s = \frac{2}{3} v_{as} - \frac{1}{3} v_{bs} - \frac{1}{3} v_{cs} \quad (7)$$

$$v_d^s = \frac{1}{\sqrt{3}} (v_{cs} - v_{bs}) \quad (8)$$

$$v_0^s = \frac{1}{3} (v_{as} + v_{bs} + v_{cs}) \quad (9)$$

The superscript 's' is used to denote that the reference frame is fixed in the stator[10]. Substituting Eqs. 4-6 in Eqs. 7-9

$$v_q^s = \frac{2}{3} e_{ag} - \frac{1}{3} e_{bg} - \frac{1}{3} e_{cg} \quad (10)$$

$$v_d^s = \frac{1}{\sqrt{3}} (e_{cg} - e_{bs}) \quad (11)$$

$$v_{0g}^s = \frac{1}{3} (e_{ag} + e_{bg} + e_{cg}) \quad (12)$$

where

$$v_{0g}^{s} = v_{0}^{s} + v_{sg} \tag{13}$$

Because the system is linear the d-q-0 voltages expressed in a stationary frame are of the same frequency as the source. Hence

$$v_{q}^{s} = V_{q\alpha}\cos\omega_{e}t + V_{q\gamma}\sin\omega_{e}t \tag{14}$$

$$v_{d}^{s} = V_{d\alpha}\cos\omega_{e}t + V_{d\gamma}\sin\omega_{e}t \tag{15}$$

$$v_{0g}^{s} = V_{0\alpha}\cos\omega_{e}t + V_{0\gamma}\sin\omega_{e}t \tag{16}$$

Equations 10-12 imply that

$$V_{q\alpha} = \frac{2}{3}E_{a\alpha} - \frac{1}{3}E_{b\alpha} - \frac{1}{3}E_{c\alpha} \tag{17}$$

$$V_{q\gamma} = \frac{2}{3}E_{a\gamma} - \frac{1}{3}E_{b\gamma} - \frac{1}{3}E_{c\gamma} \tag{18}$$

$$V_{d\alpha} = \frac{1}{\sqrt{3}}(E_{c\alpha} - E_{b\alpha}) \tag{19}$$

$$V_{d\gamma} = \frac{1}{\sqrt{3}}(E_{c\gamma} - E_{b\gamma}) \tag{20}$$

$$V_{0\alpha} = \frac{1}{3}(E_{a\alpha} + E_{b\alpha} + E_{c\alpha}) \tag{21}$$

$$V_{0\gamma} = \frac{1}{3}(E_{a\gamma} + E_{b\gamma} + E_{c\gamma}) \tag{22}$$

Since the inductances of a synchronous machine become time invariant only when expressed in the rotor reference frame, the stator voltages must also be expressed in this frame of reference. The equations of transformation from the d-q stationary to d-q rotor reference frame is illustrated by Fig. 2 and expressed analytically as

$$v_{q}^{r} = v_{q}^{s}\cos(\omega_{e}t + \delta) - v_{d}^{s}\sin(\omega_{e}t + \delta) \tag{23}$$

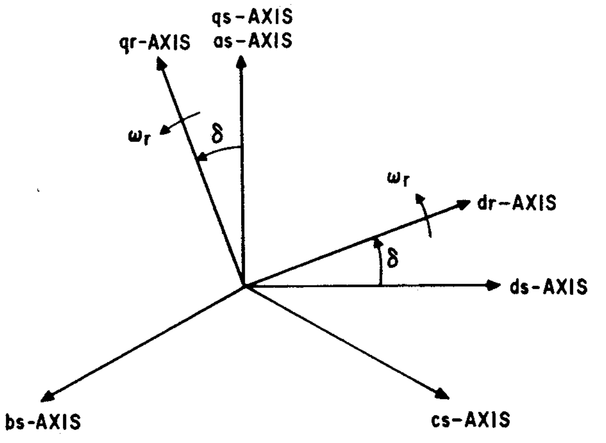

Fig. 2. Reference frame orientation at t=0.

$$v_d^r = v_q^s \sin(\omega_e t + \delta) + v_d^s \cos(\omega_e t + \delta) \quad (24)$$

$$v_{0g}^r = v_{0g}^s \quad (25)$$

In Eqs. 23-25, it has already been assumed that the rotor is synchronously rotating ($\omega_r = \omega_e$). The quantity δ is the alignment of the rotating q-axis with respect to the stationary q-axis (as axis) at t=0. The superscript 'r' denotes d-q quantities in the rotor reference frame. When the stator phase voltages are a balanced cosinusoidal set, i.e., when $v_{as} = V \cos\omega_e t$, etc., then δ corresponds to the conventional torque angle. Since the counterclockwise direction is considered positive, δ is positive for generator action.

Upon performing the above indicated transformation of variables the stator voltages can be expressed in the rotor reference frame as

$$v_q^r = \frac{1}{2}(V_{q\alpha} - V_{d\gamma})\cos\delta - \frac{1}{2}(V_{q\gamma} + V_{d\alpha})\sin\delta + \frac{1}{2}(V_{q\alpha} + V_{d\gamma})\cos(2\omega_e t + \delta) + \frac{1}{2}(V_{q\gamma} - V_{d\alpha})\sin(2\omega_e t + \delta) \quad (26)$$

$$v_d^r = \frac{1}{2}(V_{q\alpha} - V_{d\gamma})\sin\delta + \frac{1}{2}(V_{q\gamma} + V_{d\alpha})\cos\delta + \frac{1}{2}(V_{q\alpha} + V_{d\gamma})\sin(2\omega_e t + \delta) - \frac{1}{2}(V_{q\gamma} - V_{d\alpha})\cos(2\omega_e t + \delta) \quad (27)$$

$$v_{0g}^r = V_{0\alpha}\cos\omega_e t + V_{0\gamma}\sin\omega_e t \quad (28)$$

SYSTEM EQUATIONS IN MATRIX FORM

The stator voltages as defined by Eqs. 26-28 can now be "applied" to Park's Equations. These equations which define operation of a synchronous machine in a rotor reference frame are summarized in Appendix A. By convention, the superscript 'r' is generally omitted when writing Park's Equations since the rotor reference frame is implicit in their formulation. Using the motor conventions for the stator currents, Park's Equations in matrix form (without the superscript 'r') can be written in per unit as

$$\begin{bmatrix} v_q \\ v_d \\ v_{0g} \\ 0 \\ 0 \\ e_x \end{bmatrix} = \begin{bmatrix} r_a & \frac{\omega_e}{\omega_b}X_d & 0 & 0 & \frac{\omega_e}{\omega_b}X_{ad} & \frac{\omega_e}{\omega_b}X_{ad} \\ -\frac{\omega_e}{\omega_b}X_q & r_a & 0 & -\frac{\omega_e}{\omega_b}X_{aq} & 0 & 0 \\ 0 & 0 & r_a+r_g & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{kq} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{kd} & 0 \\ 0 & 0 & 0 & 0 & 0 & X_{ad} \end{bmatrix} \times \begin{bmatrix} i_q \\ i_d \\ i_{0g} \\ i_{kq} \\ i_{kd} \\ i_{fd} \end{bmatrix} \quad (29)$$

$$+\frac{p}{\omega_b}\begin{bmatrix} X_q & 0 & 0 & X_{aq} & 0 & 0 \\ 0 & X_d & 0 & 0 & X_{ad} & X_{ad} \\ 0 & 0 & X_{\ell a}+X_g & 0 & 0 & 0 \\ X_{aq} & 0 & 0 & X_{kq} & 0 & 0 \\ 0 & X_{ad} & 0 & 0 & X_{kd} & X_{ad} \\ 0 & X_{ad}^2/r_{fd} & 0 & 0 & X_{ad}^2/r_{fd} & X_{fd}X_{ad}/r_{fd} \end{bmatrix} \times \begin{bmatrix} i_q \\ i_d \\ i_{0g} \\ i_{kq} \\ i_{kd} \\ i_{fd} \end{bmatrix} \quad (29)$$

Equation 29 is equivalent to the matrix equation

$$\bar{v} = (\bar{R}+\bar{G})\bar{i} + \frac{p}{\omega_b}\bar{X}\,\bar{i} \quad (30)$$

where $\bar{v}$ and $\bar{i}$ are 6×1 voltage and current vectors, $\bar{R}$ and $\bar{G}$ are 6×6 matrices corresponding respectively to the diagonal resistive elements and the off-diagonal "speed voltage" terms of Eq. 29, and $\bar{X}$ is the 6×6 reactance matrix given in Eq. 29.

It is assumed that before the transient occurs, only DC terms appear in the vector $\bar{v}$ (balanced conditions) so that the initial condition corresponding to Eq. 30 is found by setting $p\bar{i} = 0$. Hence

$$\bar{i}(0) = (\bar{R}+\bar{G})^{-1}\,\bar{v}(0) \quad (31)$$

where $\bar{v}(0)$ is the voltage vector containing only the constant terms which prevailed before the disturbance.

After the disturbance occurs the stator voltages v_q, v_d and v_0 can contain components having zero frequency (DC), synchronous angular frequency (ω_e), and twice synchronous angular frequency ($2\omega_e$). In order to arrive at a compact solution, it is convenient to define auxiliary variables to express the sinusoidal part of the input voltages. If $x_{2\alpha}$ and $x_{2\gamma}$ are defined such that

$$x_{2\alpha} = \cos(2\omega_e t + \delta) \quad (32)$$

$$x_{2\gamma} = \sin(2\omega_e t + \delta) \quad (33)$$

then $x_{2\alpha}$ and $x_{2\gamma}$ can be obtained from the set of differential equations

$$\frac{p}{\omega_b}x_{2\alpha} = -\frac{2\omega_e}{\omega_b}x_{2\gamma} \quad (34)$$

$$\frac{p}{\omega_b}x_{2\gamma} = \frac{2\omega_e}{\omega_b}x_{2\alpha} \quad (35)$$

where

$$x_{2\alpha}(0) = \cos\delta \quad (36)$$

$$x_{2\gamma}(0) = \sin\delta \quad (37)$$

In matrix form Eqs. 34 and 35 can be written

$$\frac{p}{\omega_b}\begin{bmatrix} x_{2\alpha} \\ x_{2\alpha} \end{bmatrix} = \begin{bmatrix} 0 & -2\omega_e/\omega_b \\ 2\omega_e/\omega_b & 0 \end{bmatrix}\begin{bmatrix} x_{2\alpha} \\ x_{2\gamma} \end{bmatrix} \quad (38)$$

or simply

$$\frac{p}{\omega_b}\bar{x}_2 = \bar{\Omega}_2\bar{x}_2 \quad (39)$$

where

$$\bar{x}_2(0) = [\cos\delta,\ \sin\delta]^t \quad (40)$$

In a similar manner if $x_{1\alpha} = \cos\,\omega_e t$, $x_{1\gamma} = \sin\,\omega_e t$ then

$$\frac{p}{\omega_b}\bar{x}_1 = \bar{\Omega}_1\bar{x}_1 \quad (41)$$

where

$$\bar{x}_1(0) = [1,0]^t \quad (42)$$

The quantities $\bar{x}_1$ and $\bar{\Omega}_1$ are defined similar to $\bar{x}_2$ and $\bar{\Omega}_2$.

The voltage vector $\bar{v}$ can now be written as the sum of three components corresponding to the DC, ω_e and $2\omega_e$ components in the terminal voltage,

$$\bar{v} = \bar{v}_0 + \bar{v}_1 + \bar{v}_2 \quad (43)$$

The three voltage components are given by the vectors

$$\bar{v}_0 = \begin{bmatrix} \frac{1}{2}(V_{q\alpha}-V_{d\gamma})\cos\delta - \frac{1}{2}(V_{q\gamma}+V_{d\alpha})\sin\delta \\ \frac{1}{2}(V_{q\alpha}-V_{d\gamma})\sin\delta + \frac{1}{2}(V_{q\gamma}+V_{d\alpha})\cos\delta \\ 0 \\ 0 \\ 0 \\ E_x \end{bmatrix} \quad (44)$$

and

$$\bar{v}_1 = \bar{C}_1\bar{x}_1 \quad (45)$$

where

$$\bar{C}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ V_{0\alpha} & V_{0\gamma} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (46)$$

and

$$\bar{v}_2 = \bar{C}_2\bar{x}_2 \tag{47}$$

where

$$\bar{C}_2 = \begin{bmatrix} \frac{1}{2}(V_{q\alpha}+V_{d\gamma}) & \frac{1}{2}(V_{q\gamma}-V_{d\alpha}) \\ \frac{1}{2}(V_{q\gamma}-V_{d\alpha}) & \frac{1}{2}(V_{q\alpha}+V_{d\gamma}) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{48}$$

Substituting Eqs. 43-48 into Eq. 30, and rearranging into state variable form

$$p\bar{i} = \omega_b X^{-1}[-(\bar{R}+\bar{G})\bar{i} + \bar{C}_2\bar{x}_2 + \bar{C}_1\bar{x}_1 + \bar{v}_0] \tag{49}$$

Combining Eqs. 39 and 41 with Eq. 49 the entire system can now be represented by the composite state variable matrix equation

$$p\begin{bmatrix} \bar{i} \\ \bar{x}_2 \\ \bar{x}_1 \end{bmatrix} = \omega_b \begin{bmatrix} -\bar{X}^{-1}(\bar{R}+\bar{G}) & \bar{X}^{-1}\bar{C}_2 & \bar{X}^{-1}\bar{C}_1 \\ \bar{0}_{2\times6} & \bar{\Omega}_2 & \bar{0}_{2\times2} \\ \bar{0}_{2\times6} & \bar{0}_{2\times2} & \bar{\Omega}_1 \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{x}_2 \\ \bar{x}_1 \end{bmatrix} + \begin{bmatrix} \omega_b\bar{X}^{-1} \\ \bar{0}_{2\times6} \\ \bar{0}_{2\times6} \end{bmatrix} \bar{v}_0 \tag{50}$$

where $\bar{0}_{nxm}$ is a nxm matrix of zeros.

Equation 50 is equivalent to the state variable matrix equation

$$p\bar{x} = \bar{A}\bar{x} + \bar{u} \tag{51}$$

where $\bar{x}$ is the state vector given by

$$\bar{x} = \begin{bmatrix} \bar{i} \\ \bar{x}_2 \\ \bar{x}_1 \end{bmatrix} \tag{52}$$

The matrix $\bar{A}$ is the 10×10 system matrix of Eq. 50 and $\bar{u}$ is the 10×1 input vector of constant elements

$$\bar{u} = \begin{bmatrix} \bar{\omega}_b\bar{X}^{-1}\bar{v}_0 \\ \bar{0}_{4\times1} \end{bmatrix} \tag{53}$$

The system equations represented by Eq. 50 are now in a form amenable to a state variable solution. In particular, note that the input excitation vector u is constant so that this equation can be integrated directly without difficulty. Using modal theory the complete solution can then be expressed directly in terms of the eigenvalues and eigenvectors of the system. This solution is given in Appendix C. It is further demonstrated therein how this result can be expanded such that the time domain expression for all state variables can be written simply as a weighted series of exponentials. The expansion method described in Appendix C is readily implemented with a digital computer. Since the result is general, this formulation provides a convenient means for developing the solution to a wide variety of transient conditions.

COMPARISON WITH CONVENTIONAL METHODS

One of the most studied and best understood machine transient is the three-phase short circuit. It is of interest to use this well known transient to compare the newly developed theory with conventional techniques. The solution for a three-phase fault is readily obtained from the general solution simply by equating the source voltages e_{ag}, e_{bg}, and e_{cg} to zero (see Fig. 1). For simplicity the solution is usually given for a short circuit on the machine terminals with the machine at no load (δ=0). The computer printout for this condition is shown in Fig. 3 using the parameters of a 30 MW turbogenerator given in Appendix B. Note that the solution has been presented so that it can be read directly in mathematical form without difficulty. Although only the d-q currents are shown, solutions for the machine flux linkages can also be outputted if desired since these variables are merely algebraic combinations of system state variables (machine currents). The time domain expression for electromagnetic torque can also be calculated by means of Eq. A-7. It should be emphasized that solutions for these additional machine variables are rarely given in the literature even for the well documented condition of a three-phase fault.

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THE SOLUTION FOR I Q  IS:
  -0.4794E-03
+  0.5583E+01*EXP(-0.3595E+01*T)*SIN(0.3141E+03*T -0.1798E+03)
  -0.4074E-01*EXP(-0.9544E+01*T)
+  0.5988E-01*EXP(-0.5538E+01*T)
+  0.5510E-02*EXP(-0.1069E+01*T)

THE SOLUTION FOR I D  IS:
  -0.5000E+00
+  0.5863E+01*EXP(-0.3595E+01*T)*SIN(0.3141E+03*T+ 0.8995E+02)
  -0.1621E+01*EXP(-0.9544E+01*T)
  -0.3485E-03*EXP(-0.5538E+01*T)
  -0.3741E+01*EXP(-0.1069E+01*T)

THE SOLUTION FOR I KQ IS:
  -0.2213E-04
+  0.5466E+01*EXP(-0.3595E+01*T)*SIN(0.3141E+03*T+ 0.3385E+00)
+  0.4207E-01*EXP(-0.9544E+01*T)
  -0.6439E-01*EXP(-0.5538E+01*T)
  -0.9947E-02*EXP(-0.1069E+01*T)

THE SOLUTION FOR I KD IS:
  -0.5849E-06
+  0.4485E+01*EXP(-0.3595E+01*T)*SIN(0.3141E+03*T -0.8913E+02)
+  0.3863E+01*EXP(-0.9544E+01*T)
  -0.5508E-03*EXP(-0.5538E+01*T)
+  0.6220E+00*EXP(-0.1069E+01*T)

THE SOLUTION FOR I FD IS:
   0.5376E+00
+  0.1284E+01*EXP(-0.3595E+01*T)*SIN(0.3141E+03*T -0.9301E+02)
  -0.2119E+01*EXP(-0.9544E+01*T)
+  0.8611E-03*EXP(-0.5538E+01*T)
+  0.3401E+01*EXP(-0.1069E+01*T)
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Fig. 3. Computer printout for a three-phase fault from no-load.

The corresponding solutions for the armature direct and quadrature axis currents and field winding current obtained from the literature[1,2] are given by Eqs. 54-56.

$$i_d = -\frac{V}{X_d} + V\left(\frac{1}{X_d} - \frac{1}{X_d'}\right)e^{-t/\tau_d'} + V\left(\frac{1}{X_d'} - \frac{1}{X_d''}\right)e^{-t/\tau_d''} + \frac{V}{X_d''}e^{-t/\tau_a}\cos\omega_e t \quad (54)$$

$$i_q = -\frac{V}{X_q''}e^{-t/\tau_a}\sin\omega_e t \quad (55)$$

$$i_{fd} = \frac{V}{X_{ad}} + \frac{V}{X_{ad}}\left(\frac{X_d}{X_d'} - 1\right)\left[e^{-t/\tau_d'} - \left(1 - \frac{\tau_{kd}}{\tau_d''}\right)e^{-t/\tau_d''} - \frac{\tau_{kd}}{\tau_d''}e^{-t/\tau_a}\cos\omega_e t\right] \quad (56)$$

where V is the magnitude of the terminal voltage before the fault. Since $\delta=0$ and the machine is unloaded, then before the fault $v_q(0^-)=V$, $v_d(0^-)=0$ and $e_x(0^-)=V$. By comparing Eqs. 54 and 55 against the solution in Fig. 3 it is possible to identify x_d, x_d', x_d'', x_q'' and the time constants τ_a, τ_d', τ_d'' and τ_q''. The values thus obtained are shown in Table 1 along with those calculated using traditional methods[2]. The eigenvalue solution and standard solution for i_q, i_d and i_{fd} are compared term-by-term in Table 2. For purposes of comparison, the machine current components are defined according to the equation:

$$i_x = i_{xss} + i_{xa}\sin(\omega_e t+\theta)e^{-t/\tau_a} + i_{xd}'e^{-t/\tau_d'} + i_{xd}''e^{-t/\tau_d''} + i_{xq}''e^{-t/\tau_q''} \quad (57)$$

where x = d, q or f. It can be observed that the agreement between the two methods is very good. Slight differences between the two sets of numbers are due to approximations made in the derivation of traditional equations. Such approximations include neglecting the circuit resistances when calculating the transient reactances from the complex operations impedances.

Table 1. Comparison of Transient Parameters of a 30 MW Turbogenerator Calculated with Traditional Method and Modal Analysis (Short Circuit from No-Load).

Parameter	Concordia[1]	Modal Analysis	Parameter	Concordia[1]	Modal Analysis
X_d	2.0	2.0	τ_a	0.278	0.278
X_d'	0.27	0.236	τ_d'	0.860	0.935
X_d''	0.171	0.171	τ_d''	0.114	0.105
X_q''	0.179	0.179	τ_q''	0.181	0.181

It has been noted that since the solution has been formulated for an arbitrary voltage change nearly any conventional type of transient associated with one machine system can be solved by specifying the appropriate source voltages before and after the disturbance. As a typical example of flexibility of the solution method, Fig. 4 shows the result obtained for a line-to-ground fault wherein $e_{ag}=0$. Again the 30 MW generator of Appendix B was assumed. However, in this case the machine has been loaded to rated input power. The machine is operating with 0.926 lagging power factor at the machine terminals before the

Table 2. Comparison of Transient Currents of a 30 MW Turbogenerator Calculated with Traditional Method and with Modal Analysis (Short Circuit from No-Load)

Current	Components	Concordia[1]	Modal Analysis
i_d	i_{dss}	-0.5	-0.5
	i_{da}	5.85	5.86
	i'_{dd}	-3.20	-3.74
	i''_{dd}	-2.14	-1.62
i_{fd}	i_{fss}	0.54	0.54
	i_{fa}	1.17	1.28
	i'_{fd}	3.44	3.40
	i''_{fd}	-2.27	-2.12
i_q	i_{qss}	0.0	0.0
	i_{qa}	5.58	5.58
	i'_{qd}	0.0	0.006
	i''_{qd}	0.0	-0.041
	i''_{qq}	0.0	0.060

fault. A three-wire machine connection is assumed ($r_g=x_g=\infty$). Neglecting second order terms the solution obtained from expansion of the modal solution is

$$i_q = -0.277+1.86\sin(2\omega_e t+125°)+3.06e^{-3.6t}\sin(\omega_e t-89°)+1.41e^{-5.54t}-.007e^{-9.54t}$$

$$i_d = -1.013+1.95\sin(2\omega_e t+35°)-3.21e^{-3.6t}\sin \omega_e t -0.29e^{-9.54t}-0.69e^{-1.07t}$$

$$i_{kq} = 1.82\sin(2\omega_e t-54.8°)+3.0e^{-3.6t}\sin(\omega_e t+91°)-1.51e^{-5.54t}$$

$$i_{kd} = 1.49\sin(2\omega_e t-145°)+2.46e^{-3.6t}\sin(\omega_e t+1.5°)+0.70e^{-9.54t}+0.115e^{-1.07t} -0.013e^{-5.54t}$$

$$i_{fd} = 1.29+0.43\sin(2\omega_e t-146.5°)+0.70e^{-3.6t}\sin(\omega_e t-2.4°)-0.38e^{-9.54t} +0.628e^{-1.07t}+0.02e^{-5.54t}$$

These functions have been plotted in Fig. 4 together with the instantaneous torque. The presence of the decaying 50 Hz transient and steady-state 100 Hz terms are clearly evident.

For purposes of comparison, the same machine transient was computed using an established digital computer simulation program which uses a simple Runge-Kutta two-step integration algorithm. In this case the exact nonlinear equations

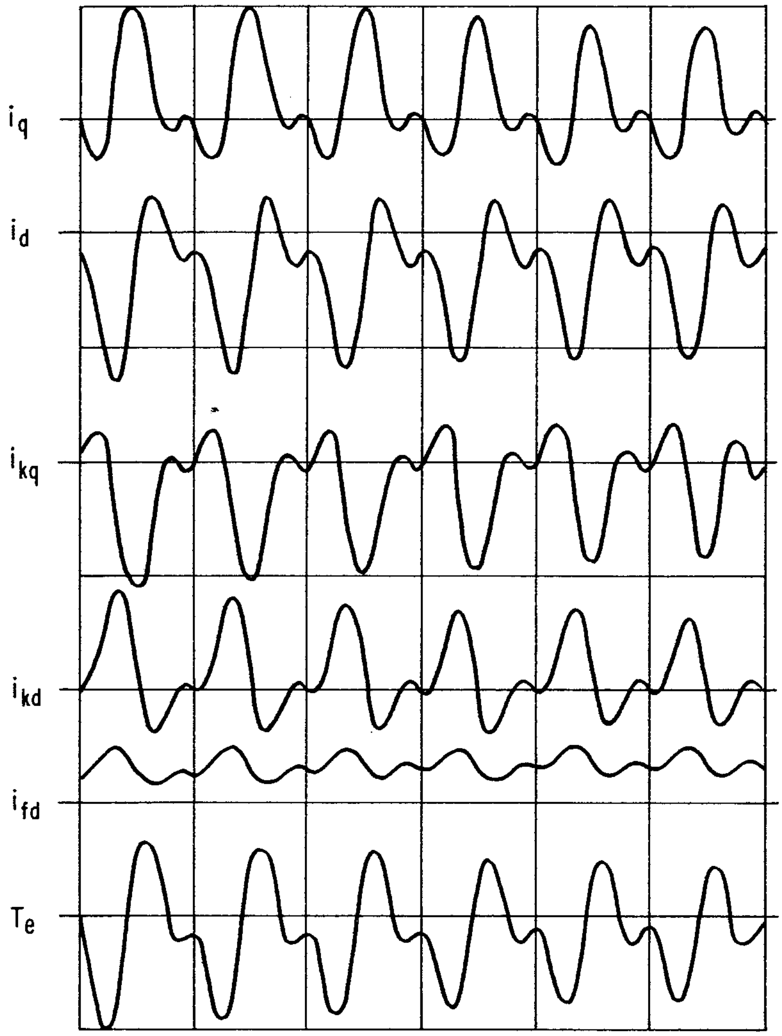

Fig. 4. Machine currents for single-phase fault from rated load. Scale: Vertical: 5.0 p.u./div., Horizontal: .02 sec/div.

were solved and the machine was permitted to swing after the fault occurred. Over the 0.12 second time interval plotted in Fig. 4 the load angle δ increases only 3° from 56° to 59°. Hence, the solution is essentially superimposed on the result calculated by the modal theory. The maximum deviation of any variable from the modal theory was 0.1 p.u. during the 0.12 second interval shown.

Solutions for most other transient conditions can be handled by specifying the proper values of e_{ag}, e_{bg}, e_{cg}. It should be noted, however, that transients resulting in unbalanced line impedances such as single phase open circuits are specifically excluded since they result in extra time dependent terms in Park's Equations.

CONCLUSION

A new method for the transient analysis of synchronous machines based on modal analysis has been presented. When the winding voltages remain explicitly known after the transient occurs this method appears to offer a simple yet powerful means for calculating machine transients (based on constant speed). Since the approach is general, a wide variety of transient conditions can be investigated by simply specifying the appropriate bus voltages which apply before and after the disturbance. The resulting expressions are inherently more accurate than former methods since the effect of winding resistances are exactly accounted for. Rotor circuits of any complexity can be routinely incorporated into the analysis. If desired, transient torques, flux linkages and copper losses in any portion of the machine can be solved from the known machine currents.

Transient currents flowing in both stator and rotor circuits following a three-phase short-circuit were calculated by this method. Solutions for a single phase fault from a loaded operating condition was generated to illustrate the flexibility of the approach. The results obtained from modal analysis were shown to compare favorably with the traditional analysis techniques.

LIST OF SYMBOLS

i_d, i_q	Direct and quadrature axis current in rotor reference frame
i_{kd}, i_{kq}	Direct and quadrature axis amortisseur winding currents
p	d/dt
r_a	Resistance of armature winding
r_{fd}	Resistance of field winding
r_{kd}, r_{kq}	Resistance of direct and quadrature axis amortisseur windings
v_d, v_q	Direct and quadrature axis voltages in rotor reference frame
v_{fd}, i_{fd}	Field winding voltage and current
X_d, X_d', X_d''	Synchronous, transient, and subtransient direct axis reactances
X_{ad}, X_{aq}	Direct and quadrature axis magnetizing reactance
$X_{\ell a}$	Armature leakage reactance
X_{fd}	Self-reactance of field winding
X_{kd}, X_{kq}	Self-reactance of direct and quadrature axis amortisseur windings
X_g, r_g	Reactance and resistance of grounding reactor (or resistor)
X_q, X_q''	Synchronous and subtransient quadrature axis reactances
τ_a	Armature short circuit time constant
τ_d', τ_d''; τ_{do}', τ_{do}''	Direct axis transient and subtransient short circuit and open circuit time constants
τ_{kd}	Direct axis amortisseur winding time constant

τ''_q, τ''_{qo}	Quadrature axis subtransient short circuit and open circuit time constants
ψ_d, ψ_q	Armature winding flux linkages along direct and quadrature axes
ψ_{fd}	Field winding flux linkages
ψ_{kd}, ψ_{kq}	Amortisseur winding flux linkages along direct and quadrature axes
ω_e	Synchronous angular frequency
ω_b	Base angular frequency
ω_r	Rotor equivalent electrical angular frequency

REFERENCES

1. C. Concordia, Synchronous Machines - Theory and Performance, John Wiley and Sons, Inc., New York, New York, 1951.

2. B. Adkins, The General Theory of Electrical Machines, John Wiley and Sons, Inc., New York, New York, 1957.

3. R.P. Schultz, W.D. Jones, and D.N. Ewart, "Dynamic Models of Turbine Generators Derived from Solid Rotor Equivalent Circuits," IEEE Trans. Power App. Syst., Vol. PAS-92, pp. 926-933, May/June 1973.

4. K. Ogata, State Space Analysis of Control Systems, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1967.

5. D.H. Baker, P.C. Krause, P.A. Rusche, "An Investigation of Excitation System Interaction," IEEE Trans. Power App. Syst., Vol. PAS-94, pp. 705-715, May/June 1975.

6. B.T. Ooi and T.H. Barton, "The Modal Approach to Induction Motor Dynamics - Part I - Constant Speed Transients," 1972 IEEE PES Winter Power Meeting Paper C 72 045-8.

7. W. Charlton, "The Short-Circuit Modes of a Synchronous Machine," IEEE Trans. on Education, Vol. E-13, No. 1, pp. 41-44, July 1970.

8. T.A. Lipo, "Performance Calculations of a Reluctance Motor Drive by DQ Harmonic Balance," 1976 IEEE/IAS Annual Meeting Conference Record, pp. 558-567, October 11-14, 1976

9. P.C. Krause, "Method of Multiple Reference Frames Applied to the Analysis of Symmetrical Induction Machinery," IEEE Trans. Power App. Syst., Vol PAS-87, pp. 218-227, January 1968.

10. P.C. Krause and C.H. Thomas, "Simulation of Symmetrical Induction Machinery," IEEE Trans. Power App. Syst., Vol. PAS-84, pp. 1025-1037, November 1965.

11. G. Shackshaft, "General Purpose Turboalternator Model," IEE Proc., Vol. 110, No. 4, pp. 703-713, April 1963.

APPENDIX A

The per unit equations which describe the behavior of a synchronous machine (Park's Equations) may be expressed for motor action by

$$v_q = \frac{p}{\omega_b}\psi_q + \psi_d \frac{\omega_r}{\omega_b} + r_a i_q \quad \text{(A-1)}$$

$$v_d = \frac{p}{\omega_b}\psi_d - \psi_q \frac{\omega_r}{\omega_b} + r_a i_d \quad \text{(A-2)}$$

$$v_0 = \frac{p}{\omega_b}\psi_0 + r_a i_0 \quad \text{(A-3)}$$

$$0 = \frac{p}{\omega_b}\psi_{kq} + r_{kq} i_{kq} \quad \text{(A-4)}$$

$$0 = \frac{p}{\omega_b}\psi_{kd} + r_{kd} i_{kd} \quad \text{(A-5)}$$

$$e_x = \frac{X_{ad}}{r_{fd}}\left(\frac{p}{\omega_b}\psi_{fd} + r_{fd} i_{fd}\right) \quad \text{(A-6)}$$

The per unit torque, positive for motor action is expressed

$$T_e = \psi_d i_q - \psi_q i_d \quad \text{(A-7)}$$

In the above equations, ω_b is the base electrical angular velocity, ω_r is the electrical angular velocity of the rotor, and p is the operator d/dt, also

$$\psi_q = X_q i_q + X_{aq} i_{kq} \quad \text{(A-8)}$$

$$\psi_d = X_d i_d + X_{ad}(i_{kd} + i_{fd}) \quad \text{(A-9)}$$

$$\psi_0 = X_{\ell a} i_0 \quad \text{(A-10)}$$

$$\psi_{kq} = X_{kq} i_{kq} + X_{aq} i_q \quad \text{(A-11)}$$

$$\psi_{kd} = X_{kd} i_{kd} + X_{ad}(i_d + i_{fd}) \quad \text{(A-12)}$$

$$\psi_{fd} = X_{fd} i_{fd} + X_{ad}(i_d + i_{kd}) \quad \text{(A-13)}$$

APPENDIX B

The machine used for this study is a 30 MW, 11.86 kV, 50 Hz turbogenerator[11]. The per unit parameters of this machine on a 37.5 MVA base are as follows:

r_a	= 0.002	r_{fd}	= 0.001
$X_{\ell a}$	= 0.14	X_{fd}	= 2.0
$r_{kq} = r_{kd}$	= 0.003	X_{aq}	= 1.86
$X_{kq} = X_{kd}$	= 1.9	X_{ad}	= 1.86
τ'_{do}	= 6.37 s.	τ'_d	= 0.86 s.
τ''_d	= 0.114 s.	τ''_q	= 0.181 s.
H	= 2.65 s.		

APPENDIX C

REDUCTION TO EXPLICIT FORM

Since the forcing function to Eq. 50 is a vector of constants, the solution can be found by direct integration. It is well known that the solution to Eq. 50 is[4]

$$\bar{x}(t) = \bar{P}\bar{Q}(t)\bar{R}\bar{x}(0) + \bar{\Lambda}^{-1}\bar{P}[\bar{Q}(t)-\bar{I}]\bar{R}\bar{u} \tag{C-1}$$

where

$$\bar{P} = [\bar{p}_1, \bar{p}_2, \bar{p}_3, . . ., \bar{p}_m] \tag{C-2}$$

$$\bar{R} = \bar{P}^{-1} \tag{C-3}$$

$$\bar{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & . & . & 0 \\ 0 & \lambda_2 & & & . \\ . & & \ddots & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & \lambda_m \end{bmatrix} \tag{C-4}$$

$$\bar{Q}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & . & . & 0 \\ 0 & e^{\lambda_2 t} & . & & 0 \\ . & & & . & . \\ 0 & . & . & 0 & e^{\lambda_m t} \end{bmatrix} \tag{C-5}$$

The quantity $\bar{P}$ is the matrix of eigenvectors $\bar{p}_1, \bar{p}_2, . . ., \bar{p}_m$ corresponding to the eigenvalues $\lambda_1, \lambda_2, . . ., \lambda_m$ contained in the eigenvalue matrix $\bar{\Lambda}$. The integer m is the order of the system (rank of $\bar{A}$). In this application m=n+4 where n is the order of the synchronous machine equations. For a single amortisseur winding on each axis n=6. The matrix $\bar{Q}(t)$ is the response matrix of the uncoupled system. It will be assumed that no repeated eigenvalues occur so that both $\bar{\Lambda}$ and $\bar{Q}(t)$ are diagonal. The quantity $\bar{I}$ is the identity matrix.

The first term of Eq. C-1 can be recognized as the homogeneous part of the solution resulting from the initial conditions at t=0. The second part of Eq. C-1 corresponds to the complementary solution resulting from the excitation $\bar{u}$. The first term of Eq. C-1 can be written

$$\bar{P}\bar{Q}(t)\bar{R}\bar{x}(0) = \bar{P}\bar{Q}(t) \begin{bmatrix} \bar{r}_1 \cdot \bar{x}(0) \\ \bar{r}_2 \cdot \bar{x}(0) \\ . \\ . \\ \bar{r}_m \cdot \bar{x}(0) \end{bmatrix} \tag{C-6}$$

where $\bar{a}\cdot\bar{b}$ is the dot (inner) product of $\bar{a}$ and $\bar{b}$. and $\bar{r}_1$, $\bar{r}_2$ . . . $\bar{r}_m$ are the rows of matrix $\bar{R}$.

Multiplying the response matrix $\bar{Q}(t)$ by $\overline{Rx}(0)$, the first term in Eq. C-1 can be written

$$\bar{P}\bar{Q}(t)\bar{R}\bar{x}(0) = [\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_m] \begin{bmatrix} e^{\lambda_1 t}\bar{r}_1\cdot\bar{x}(0) \\ e^{\lambda_2 t}\bar{r}_2\cdot\bar{x}(0) \\ \cdot \\ \cdot \\ \cdot \\ e^{\lambda_m t}\bar{r}_m\cdot\bar{x}(0) \end{bmatrix} \tag{C-7}$$

Eq. C-7 can be written in the alternative form

$$\bar{P}\bar{Q}(t)\bar{R}\bar{x}(0) = \left\{[\bar{r}_1\cdot\bar{x}(0)]\bar{p}_1, [\bar{r}_2\cdot\bar{x}(0)]\bar{p}_2, \ldots, [\bar{r}_m\cdot\bar{x}(0)]\bar{p}_m\right\} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \cdot \\ \cdot \\ \cdot \\ e^{\lambda_m t} \end{bmatrix} \tag{C-8}$$

Equation C-8 is of the form

$$\bar{P}\bar{Q}(t)\bar{R}\bar{x}(0) = \bar{S}\bar{q} \tag{C-9}$$

where

$$\bar{q} = [e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_m t}]^t \tag{C-10}$$

Using the same manipulation technique it can be shown that the second term of Eq. C-1 can be reduced to

$$\bar{\Lambda}^{-1}\bar{P}[\bar{Q}(t)-\bar{I}]\bar{R}\bar{u} = - \begin{bmatrix} p_{11}\bar{r}_1\cdot\bar{u}/\lambda_1 & p_{12}\bar{r}_2\cdot\bar{u}/\lambda_1 & \cdots & p_{1m}\bar{r}_m\cdot\bar{u}/\lambda_1 \\ p_{21}\bar{r}_1\cdot\bar{u}/\lambda_2 & p_{22}\bar{r}_2\cdot\bar{u}/\lambda_2 & \cdots & p_{2m}\bar{r}_m\cdot\bar{u}/\lambda_2 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{m1}\bar{r}_1\cdot\bar{u}/\lambda_m & p_{m2}\bar{r}_2\cdot\bar{u}/\lambda_m & \cdots & p_{mm}\bar{r}_m\cdot\bar{u}/\lambda_m \end{bmatrix} \begin{bmatrix} 1-e^{\lambda_1 t} \\ 1-e^{\lambda_2 t} \\ \cdot \\ \cdot \\ \cdot \\ 1-e^{\lambda_m t} \end{bmatrix} \tag{C-11}$$

Equation C-11 can be expressed as

$$\bar{\Lambda}^{-1}\bar{P}[\bar{Q}(t)-\bar{I}]\bar{R}\bar{u} = T[\bar{1}-\bar{q}(t)] \tag{C-12}$$

where

$$\bar{1} = [1,1,1,..,1]^t \quad (m\times 1 \text{ vector}) \tag{C-13}$$

Hence the solution for x(t) given by Eq. C-1 can be written in the alternative form.

$$\bar{x}(t) = (\bar{S}-\bar{T})\bar{q}(t) + \bar{T}\bar{1} \tag{C-14}$$

It is clear that when t=0, $\bar{x}(0)=\bar{S}\bar{1}$. If the eigenvalues of $\bar{A}$ all have negative real parts, then when $t\rightarrow\infty$, $\bar{x}(\infty) \rightarrow \bar{T}\bar{1}$.

COMPLEX EIGENVALUES

When the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ are not all real, complex terms appear in the $\bar{S}$ and $\bar{T}$ matrices and it is useful to rewrite these terms in a different form. Consider first the homogeneous part of the solution. Because the roots are complex conjugates two terms in the solution will appear typically as

$$x_h = (a+jb)e^{(\sigma+j\omega)t} + (a-jb)e^{(\sigma-j\omega)t} \tag{C-15}$$

Eq. C-15 can be rearranged to the form

$$x_h = 2a\ e^{\sigma t}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) - 2b\ e^{\sigma t}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) \tag{C-16}$$

Equation C-16 is equivalent to

$$x_h = e^{\sigma t}[2a \cos\omega t - 2b \sin\omega t] \tag{C-17}$$

or alternatively

$$x_h = c\ e^{\sigma t}\sin(\omega t+\phi) \tag{C-18}$$

where

$$c = 2\sqrt{a^2 + b^2} \tag{C-19}$$

$$\phi = \tan^{-1}(a/-b) \tag{C-20}$$

In a similar manner it can be shown that two complex conjugate terms in the complementary part of the solution reduce to

$$x_c = 2a - c\ e^{\sigma t}\sin(\omega t+\phi) \tag{C-21}$$

where c and ϕ are defined as above.

Manuscript received in final form, March 8, 1977