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A Cartesian Vector Approach To Reference Frame Theory of AC Machines

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Introduction

The study of electric machinery involves, essentially, the behavior of circuits in relative motion. Hence, the topic of transformation theory or reference frame theory constitutes an essential aspect of AC machine analysis. In many texts, each type of electric machine is presented separately and a specific change of variables appropriate for each machine is introduced, typically by means of an intuitive argument rather than in a rigorous manner. Because of the highly coupled nature of the AC machine windings, the benefits of the transformation of variables is intertwined with the harmonic behavior of the machine inductances under consideration so that the essential nature of the transformation is typically lost.

In this paper it is shown that the theory concerning three phase AC systems can be considered as an extension of the three dimensional Cartesian space of engineering mechanics. Consequently, the transformation of variables to a rotating axes can be interpreted as a transformation from stationary to rotating three dimensional Cartesian axes. It is shown that the equations of transformation to these rotating axes corresponds identically to the rotating reference frame of AC machine theory. Hence, the usual d-q-0 transformation is demonstrated to be a property of a three dimensional Cartesian space and not solely a mathematical manipulation peculiar to AC Machine Theory. This new interpretation of d-q-0 variables permits the subject of transformation theory to be presented in a rigorous manner rather than as an "a priori" assumption during the development of Stanley's and Park's Equations.

A Brief Review of Motion in a Rotating Reference Frame

Many of the fundamental problems of mechanics involves the concept of particle motion or motion of a point in space. The position of the particle or point is defined by its radius vector $\mathbf{r}$, whose components are expressed in Cartesian co-ordinates as x, y, z. If the co-ordinate system is rigid and is not in motion then the reference frame is called the inertial frame. The radius vector locating the particle in the inertial frame is conveniently expressed in the form

$$\mathbf{r}^s = x^s \mathbf{u}_x^s + y^s \mathbf{u}_y^s + z^s \mathbf{u}_z^s \tag{1}$$

where $\mathbf{u}_x^s$, $\mathbf{u}_y^s$, $\mathbf{u}_z^s$ are unit vectors along the x,y and z axes respectively and the superscript "s" signifies that the reference frame is "stationary". The time derivative $\dot{\mathbf{r}} = d\mathbf{r}/dt$ is the velocity vector of the particle while the second derivative $\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$ is its acceleration vector.

Consider now a second reference frame attached to the origin of the inertial frame which exhibits a pure rotation about the inertial frame as shown in Fig. 1. The location of the particle in the rotating frame is expressed by

$$\mathbf{r}^r = x^r \mathbf{u}_x^r + y^r \mathbf{u}_y^r + z^r \mathbf{u}_z^r \tag{2}$$

where the superscript "r" expresses the fact that the reference frame is "rotating". The particle moves in the (x^r, y^r, z^r) coordinate system with the velocity

$$\dot{\mathbf{r}}^r = d\mathbf{r}^r/dt = dx^r/dt\,\mathbf{u}_x^r + dy^r/dt\,\mathbf{u}_y^r + dz^r/dt\,\mathbf{u}_z^r$$
$$+ x^r d\mathbf{u}_x^r/dt + y^r d\mathbf{u}_y^r/dt + z^r d\mathbf{u}_z^r/dt \tag{3}$$

The first three terms in Eq. 3 is simply the velocity of the point as viewed in the rotating frame while the last three terms account for the motion of the rotating frame itself. The second group of three terms can be written in a compact form by defining the instantaneous angular velocity vector

$$\omega = \omega_x \mathbf{u}_x^r + \omega_y \mathbf{u}_y^r + \omega_z \mathbf{u}_z^r \tag{4}$$

The angular velocities ω_x, ω_y, and ω_z express the angular rotation of the yz, zx and xy planes of the rotating frame about the x^s, y^s and z^s axes respectively

It is shown in texts on classical mechanics [1], that the velocity of the particle can then be written in the vector form

$$\dot{\mathbf{r}}^s = \dot{\mathbf{r}}^r + \omega \times \mathbf{r}^r \tag{5}$$

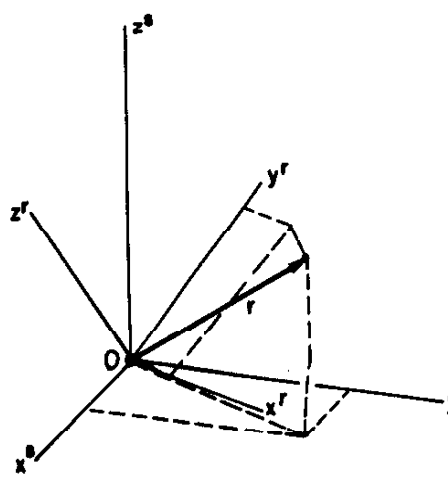

Fig. 1 Stationary and Rotating Cartesian Co-ordinate Systems.

Cartesian Vector Approach to Three Phase Circuits

While rotating co-ordinates are widely used in mechanics to study forces acting on rotating bodies, a special version of this transformation has also proven useful when the rotating device is of electrical rather than mechanical origin. The common requirement for the use of such transformations is that the device have a certain amount of physical symmetry. As the simplest possible illustration of such symmetry consider the three phase r-L network shown in Fig. 2. In general, the three source voltages and resulting currents can be arbitrary functions of time which are either explicitly known or implicitly known through the solution of the r-L circuit equations. However, it is assumed that the load is "balanced". That is, the resistance and inductance of each of the three phases are assumed to be equal.

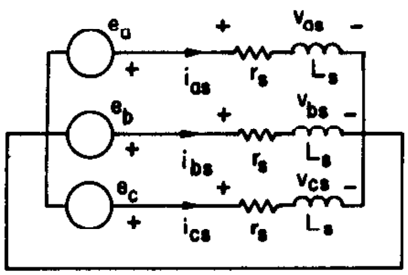

Fig. 2 Balanced Three Phase r-L Circuits

The concept of Cartesian vectors can be readily extended to the solution of this problem if we now define a pair of three dimensional Cartesian spaces having, in turn, the units of volts and of amperes. The three phase voltages v_{as}, v_{bs}, and v_{cs} can be considered as components of the three dimensional Cartesian "voltage space". It is important to note that since the three Cartesian axes are mutually perpendicular, this concept is entirely different than the usual space vector representation in which the three coordinate axes are positioned on a plane, displaced mutually by 120 electrical degrees[2]. The location of a point in this voltage space describing the instantaneous values of the three phase voltages can be expressed as

$$\mathbf{v}^s_{abcs} = v_{as}\,\mathbf{u}^s_{as} + v_{bs}\,\mathbf{u}^s_{bs} + v_{cs}\,\mathbf{u}^s_{cs} \tag{6}$$

In analogous manner,

$$\mathbf{i}^s_{abcs} = i_{as}\,\mathbf{u}^s_{as} + i_{bs}\,\mathbf{u}^s_{bs} + i_{cs}\,\mathbf{u}^s_{cs} \tag{7}$$

where the superscript s again signifies that the frame of reference is the inertial or stationary reference frame.

The three circuit equations can now be easily combined to form the single vector equation

$$\mathbf{v}^s_{abcs} = r_s\,\mathbf{i}^s_{abcs} + L_s\,\frac{d\,\mathbf{i}^s_{abcs}}{dt} \tag{8}$$

While the solution of such a simple problem does not require special analysis techniques, it is useful to consider the possibility of analyzing this same problem in a rotating reference frame. It is clear that from Eq. 5 that in a rotating frame, Eq. 8 becomes,

$$\mathbf{v}^r_{abcs} = r_s\,\mathbf{i}^r_{abcs} + L_s\,\frac{d\,\mathbf{i}^r_{abcs}}{dt} + \boldsymbol{\omega} \times \mathbf{i}^r_{abcs} \tag{9}$$

where the vector $\boldsymbol{\omega}$ portrays the angular velocity of the rotating frame and is perpendicular to its instantaneous plane of rotation.

It is apparent that the vectors $\mathbf{v}_{abcs}$ and $\mathbf{i}_{abcs}$ move somewhere in a three dimensional Cartesian space and if the current and voltage variables are periodic functions, then these curves are closed. However, in addition, since the load is a three wire system it follows that

$$i_{as} + i_{bs} + i_{cs} = 0 \tag{10}$$

It can be recalled, however, from principles of analytic geometry that the definition of a plane in a three dimensional space is

$$ax + by + cz = d \tag{11}$$

Hence, Eq. 10 describes a surface of a plane passing through the origin (d=0) in which the coefficients a,b,c of the plane are all unity. Consequently, the motion of the vector $\mathbf{i}_{abcs}$ always remains on a plane in the three dimensional space. Similarly, it can be readily shown that since the three loads are balanced, the sum of the three line to neutral load voltages sum to zero and, therefore, also rotates on a plane. Since the scalar coefficients a,b,c and d are the same, the voltage and current vectors rotate on the similar planes.

Plane Containing the Vectors $\mathbf{v}_{abcs}$ and $\mathbf{i}_{abcs}$ for a 3-Wire System

Since the current and voltage vectors of a 3 wire system rotate on a plane it is of interest to locate this plane by constructing a coordinate system in which the two axes of the new system is aligned in the plane of rotation. Such an alignment procedure is readily accomplished by locating a suitable pair of orthogonal unit vectors.

Since one of the two required vectors in the plane is identically either the current or the voltage vector itself it is useful to select one of the two variables as the reference quantity. If the voltage vector is chosen, the reference frame can be said to be *fixed to the voltage vector* while if current is selected the reference frame is *fixed to the current vector*.

While any voltage or current vector which is an admissible solution to Fig. 2 can be chosen as reference, the conventional transformation used in AC machine analysis utilizes the simple case in which the source voltages are assumed to be balanced and sinusoidal. The three components of the line to neutral load voltages are then given by

$$\begin{aligned} v_{as} &= V_m \cos \omega_e t \\ v_{bs} &= V_m \cos (\omega_e t - 2\pi/3) \\ v_{cs} &= V_m \cos (\omega_e t + 2\pi/3) \end{aligned} \tag{12}$$

As previously mentioned, one of the unit vectors in the plane of rotation is clearly the voltage vector itself normalized to unity. That is

$$\mathbf{u}_q^e = \frac{\mathbf{v}_{abcs}}{|\mathbf{v}_{abcs}|}$$

$$= \sqrt{\frac{3}{2}}[\mathbf{u}_{as}^s \cos \omega_e t + \mathbf{u}_{bs}^s \cos(\omega_e t - 2\pi/3) + \mathbf{u}_{cs}^s \cos(\omega_e t + 2\pi/3)] \quad (13)$$

A second unit vector in the plane of rotating can be found by locating another vector which is not co-linear with $\mathbf{u}_q^e$. In this regard, it can be observed that since the amplitude $|\mathbf{v}_{abcs}|$ is a constant, the motion of the voltage vector on the plane traces out a circular path. In this case, the instantaneous derivative of the voltage vector will always be orthogonal to the vector itself and clearly lies in the plane of rotation. Hence,

$$\mathbf{u}_d^e = -\frac{\frac{d\mathbf{v}_{abcs}}{dt}}{\left|\frac{d\mathbf{v}_{abcs}}{dt}\right|}$$

$$= \frac{1}{\sqrt{2}}[\mathbf{u}_{as}^s \sin \omega_e t + \mathbf{u}_{bs}^s \sin(\omega_e t - 2\pi/3) + \mathbf{u}_{cs}^s \sin(\omega_e t + 2\pi/3)] \quad (14)$$

The negative sign is used in the definition of $\mathbf{u}_d^e$ in order to preserve convention. It is readily verified that $\mathbf{u}_d^e$ is orthogonal to $\mathbf{u}_q^e$ since the dot product $\mathbf{u}_d^e \cdot \mathbf{u}_q^e$ is identically zero.

Since the two unit vectors are in the plane a third unit vector, orthogonal to the first two and normal to the plane of rotation, is easily obtained by taking the cross product

$$\mathbf{u}_n^e = \mathbf{u}_d^e \times \mathbf{u}_q^e \quad (15)$$

Upon substituting Eqs. 13 and 14 into Eq. 15 it is eventually found that

$$\mathbf{u}_n^e = \frac{1}{\sqrt{3}}[\mathbf{u}_{as}^s + \mathbf{u}_{bs}^s + \mathbf{u}_{cs}^s] \quad (16)$$

Note the similarity of the coefficients of $\mathbf{u}_n^e$ to the zero sequence component of conventional theory.

It can be readily verified that $\mathbf{u}_n^e$ is normal to the vector $\mathbf{v}_{abcs}$ by taking the dot product of $\mathbf{v}_{abcs}$ and $\mathbf{u}_n^e$

$$\mathbf{v}_{abcs} \cdot \mathbf{u}_n^e = \frac{1}{\sqrt{3}}[v_{as} + v_{bs} + v_{cs}]$$

$$= 0 \quad (17)$$

Note that the coefficient of Eq. 16 differs slightly from the 1/3 coefficient normally chosen for the definition of zero sequence. The "n" subscript is used here to suggest that $\mathbf{u}_n^e$ is "normal" to the d-q plane. Alternatively, "n" denotes the neutral axis which corresponds to current flow in the "neutral" of the machine in the case of a four wire load.

It is apparent that Eqs. 13, 14 and 16 effectively define the synchronously rotating reference frame of classical AC machine theory. The orientation of the $d-q-n$ unit vectors at the instant $t=0$ forms the unit vectors for the stationary reference frame. When the neutral component of voltage and current is zero, the voltage and current vectors rotate on the $d-q$ plane. Figure 3 illustrates the location of the $d-q-n$ vectors for the stationary reference frame and shows the orientation of the $d-q$ plane relative to the inertial or "phase variable" reference frame.

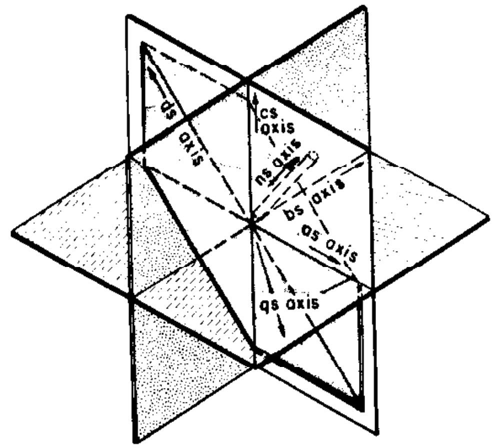

Fig. 3 Location of Stationary $d-q-n$ Co-ordinates and the $d-q$ plane in the Phase Variable Co-ordinate System.

The inverse relations to Eqs. 13, 14 and 16 are

$$\mathbf{u}_{as}^s = [\mathbf{u}_q^e \cos \omega_e t + \mathbf{u}_d^e \sin \omega_e t + \mathbf{u}_n^e/\sqrt{2}] \quad (18)$$

$$\mathbf{u}_{bs}^s = [\mathbf{u}_q^e \cos(\omega_e t - 2\pi/3) + \mathbf{u}_d^e \sin(\omega_e t + 2\pi/3) + \mathbf{u}_n^e/\sqrt{2}] \quad (19)$$

$$\mathbf{u}_{cs}^s = [\mathbf{u}_q^e \cos(\omega_e t + 2\pi/3) + \mathbf{u}_d^e \sin(\omega_e t + 2\pi/3) + \mathbf{u}_n^e/\sqrt{2}] \quad (20)$$

Equations 13,14 and 16 contain, in effect, the transformation of variables generally assumed for any real rotating or stationary reference frame. For example the transformation equations for the synchronously rotating reference frame can be obtained if we simply note that the voltage vector $\mathbf{v}$ is identical in both the phase variable and in the stationary reference frame. Hence,

$$\mathbf{v}_{abcs}^s = \mathbf{v}_{qdns}^e$$

or

$$v_{as}\mathbf{u}_{as}^s + v_{bs}\mathbf{u}_{bs}^s + v_{cs}\mathbf{u}_{cs}^s = v_{qs}^e\mathbf{u}_{qs}^e + v_{ds}^e\mathbf{u}_{ds}^e + v_{ns}^e\mathbf{u}_{ns}^e$$

Upon substituting Eqs. 13, 14 and 16 into the above, it is readily established that

$$v_{qs}^e = \sqrt{\frac{2}{3}}[v_{as}\cos \omega_e t + v_{bs}\cos(\omega_e t - 2\pi/3) + v_{cs}\cos(\omega_e t + 2\pi/3)] \quad (21)$$

$$v_{ds}^e = \sqrt{\frac{2}{3}}[v_{as}\sin \omega_e t + v_{bs}\sin(\omega_e t - 2\pi/3) + v_{cs}\sin(\omega_e t + 2\pi/3)] \quad (22)$$

$$v_{ns}^{s} = \frac{1}{\sqrt{3}}[v_{as} + v_{bs} + v_{cs}] \quad (23)$$

Note that not only are the d and q axes variables obtained from this operation but also the zero sequence component which is usually appended to the d and q components as a separate consideration [2].

The transformation equations for the stationary reference frame are obtained if we rigidly maintain the alignment taken for the unit vectors of the synchronous frame at $t = 0$. The unit vectors for the stationary frame are described by

$$\mathbf{u}_q^s = \sqrt{\frac{2}{3}}[\mathbf{u}_{as}^s - \frac{1}{2}\mathbf{u}_{bs}^s - \frac{1}{2}\mathbf{u}_{cs}^s] \quad (24)$$

$$\mathbf{u}_d^s = \frac{1}{\sqrt{2}}[\mathbf{u}_{cs}^s - \mathbf{u}_{bs}^s] \quad (25)$$

$$\mathbf{u}_{n_s} = \frac{1}{\sqrt{3}}[\mathbf{u}_{as}^s + \mathbf{u}_{bs}^s + \mathbf{u}_{cs}^s] \quad (26)$$

It can be noted that since the $\mathbf{u}_n^e$ and $\mathbf{u}_n^s$ unit vectors are both normal to the plane of rotation, they are co-linear. It is apparent that the zero sequence or normal axis unit vector will be the same for any real rotating or stationary transformation which involves balanced loads if two other unit vectors are always defined in the plane containing the voltage and current vectors for three wire connections (Eq. 10).

The corresponding transformation equations for the stationary $d-q-n$ voltage components are

$$v_{qs}^s = \sqrt{\frac{2}{3}}[v_{as} - \frac{1}{2}v_{bs} - \frac{1}{2}v_{cs}] \quad (27)$$

$$v_{ds}^s = \frac{1}{\sqrt{2}}[v_{cs} - v_{bs}] \quad (28)$$

$$v_{ns}^s = \frac{1}{\sqrt{3}}[v_{as} + v_{bs} + v_{cs}] \quad (29)$$

While a balanced sinusoidal supply has been used as aid in deducing the transformations useful in AC machine analysis, the definitions, Eqs. 13,14,16, 21-23, 24-26 and 27-29 clearly remain valid for any terminal condition, be it for unbalanced sinusoidal supplies, inverter supplies or any other type of excitation.

Scalar Form of the Vector Differential Equation

In general, the vector equation form of the three phase r-L circuit of Fig. 2 is given by Eq. 9, that is

$$\mathbf{v}_{abcs}^e = r_s \mathbf{i}_{abcs}^e + L_s \frac{d\mathbf{i}_{abcs}^e}{dt} + \omega \times \mathbf{i}_{abcs}^e \quad (30)$$

where the superscript e is used to signify the synchronously rotating reference frame.

In terms of unit vectors which have now been defined, the angular velocity vector ω is expressed by

$$\omega = \frac{d\mathbf{u}_q^e}{dt} \times \mathbf{u}_q^e \quad (31)$$

In the case of the three phase example, $\mathbf{u}_q^e$ is a unit vector directed along the position vector which in this case corresponds to $\mathbf{v}_{abcs}$. It is tedious but not difficult to show that

$$\omega = \omega_e \mathbf{u}_n^e \quad (32)$$

This result is apparent if it is recalled that the vectors $\mathbf{v}_{abcs}$ and $\mathbf{i}_{abcs}$ rotate on the $d-q$ plane and that the angular velocity vector is normal to the plane of rotation.

If Eq. 32 is substituted into Eq. 30 it can be shown that the three components of Eq. 30 are

$$v_{qs}^e = r_s i_{qs}^e + L_s \frac{di_{qs}^e}{dt} + \omega_e L_s i_{ds}^e \quad (33)$$

$$v_{ds}^e = r_s i_{ds}^e + L_s \frac{di_{ds}^e}{dt} - \omega_e L_s i_{qs}^e \quad (34)$$

$$v_{ns}^e = r_s i_{ns}^e + L_s \frac{di_{ns}^e}{dt} \quad (35)$$

These equations are the well known equations for a stationary r-L circuit in a synchronously rotating reference frame. Note that the fact that the "speed voltage" term which appears in only in the d and q axis equations results from the fact that the n axis is normal to the plane of rotation. It is clear that the "speed voltages" of rotating machine theory are simply a natural extension the concept of angular velocity of a rotating frame in engineering mechanics.

Conclusion

This paper has presented an alternative viewpoint of $d-q-0$ transformation theory which is based on the perspective of vectors in a three dimensional Cartesian space. The approach of this paper differs from the "space vector" concept of Kovacs and Racz [2] in that the three phase vectors are not co-planar but are mutually orthogonal in a three dimensional space. As a result, the transformations inherently yield the zero sequence component as well as the d and q axis components. Hence, the zero sequence component is clearly shown to be the necessary third component in the $d-q-0$ transformation and not a third equation which is simply tacked on to the $d-q$ transformation. Also, it is well known that the planar approach does not yield a power invariant transformation and that an "arbitrary constant" of $\sqrt{(2/3)}$ must be introduced rather artificially in the transformation in order to maintain power invariance. On the other hand the Cartesian vector approach inherently produces the proper transformation scale to yield power invariance. Finally, it has been shown that the issue of transformation of variables in three phase systems can be handled in a much more straightforward and understandable manner if the complexity involved with the machine inductances is temporarily put aside. The derivation then becomes a simple extension of reference frame theory in engineering mechanics with the concept of speed voltages in the circuit equations readily interpreted as relative velocity in a rotating frame.

References

[1] S.W. McCuskey,"An Introduction to Advanced Dynamics", Addison- Wesley Publishing Co. Inc., 1959.

[2] K.P. Kovacs and I. Racz,"Transient Vorgange in Wechselstrom- machinen", Verlag de Ungarishehen Akademie der Wissenshaften, Budapest 1959.