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Induction Machine Phase Balancing by Unsymmetrical Thyristor Voltage Control

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Abstract - High inrush currents and unbalanced supply voltages are two of the major causes for failure of polyphase induction machines. While thyristor voltage controllers, sometimes called static starters, are in widespread use to limit damaging inrush currents, little attention has been paid to use of these controllers to balance the phase voltages of the machine. This paper addresses technical feasibility of unsymmetric control to limit unbalanced currents. Substantial reduction in losses are shown to be possible for two specific types of unbalance.

Introduction

Unbalanced operation has long been the source of heating problems and reduced efficiency in ac induction and synchronous machines. A major cause of an unbalanced voltage at the point of utilization is typically due to single phase loads on a system which are not uniformly applied to all three phases. This is particularly true for rural electric power systems with long distribution lines but can also occur in large urban power systems where heavy single phase demands, particularly lighting loads, are imposed in large commercial facilities. A large manufacturing plant may have a well balanced incoming supply voltage, but unbalanced conditions can be developed within the plant from its own single phase power requirements if the loads are not uniformly spread among the three phases. An unbalanced condition can also be caused by an unsymmetrical transformer winding or transmission impedances, open wye, open delta, an unbalanced load in the transmission lines, and many other causes [1-4].

While the phase voltage unbalance may be small, large negative sequence currents can result due to low negative sequence impedance. These large currents, in turn, cause unbalanced heating in the machine which can potentially lead to failure. Unbalanced voltage operation will also create a pulsating torque which produces speed pulsations, mechanical vibration and consequently acoustic noise. Berndt and Schmitz [1] have shown experimentally that an unbalanced condition should be restricted to less than 5% of normal voltage if a service factor of 115% is specified. German VDE standards limit the unbalance condition to less than 2% [5].

The advent of solid state control equipment as additional loads on the distribution line, while introducing new problems such as harmonics, has also introduced new alternatives to the problem of correcting phase unbalance since such equipment is generally interposed between the source of the unbalance and the machine itself. Solid state control of induction machines can be generally relegated to two catagories, voltage control and volts per hertz control. With volts per hertz control, the flux is held constant essentially constant while the frequency is varied. Using simple voltage control, the frequency is held constant while the flux is adjusted by means of series connected pairs of SCRs.

Volts per hertz controllers, incorporating numerous types of power converters, are in widespread use in industrial applications where adjustable speed provides benefits over fixed speed operation. While a small amount of speed control is also possible, voltage controllers are generally considered inefficient for this purpose [6]. However, installation of voltage controllers are also becoming increasingly used as static motor starters in which the flux is reduced to limit the inrush starting current. In addition, voltage controllers are sometimes used as energy savers in which the flux is reduced in accordance with the connected load [7]. While often not justifiable as the sole economic consideration [8], the energy saver principle may become practical when used in conjunction with a static starter.

Conventional remedies for phase unbalance often involves costly modification of the incoming substation equipment or reworking of the feeder lines to the various loads. The problem becomes particularly difficult when the unbalance is continually varying such as with various industrial loads, for example arc furnaces loads. In such cases the possibility of using a solid state controller to correct phase unbalance appears to be particularly attractive alternative compared to other techniques such as static var regulators [9]. It appears that a certain amount of phase unbalance correction is possible by proper modification of either type of motor controller. However, with volts per hertz control, the motor is isolated from the point of the unbalance by the filtering action of the dc link so that the unbalance problem is less severe. On the other hand, the voltage controller is normally in use only during the starting period so that, in this case, the full effect of any unbalance is experienced by the motor. Since installation costs can often be justified on the basis of static starting, it is useful to consider possible benefits that might be achieved by utilization of the series connected SCRs to correct the unbalanced supply voltages. The feasibility of such a phase balancer is the subject of this paper.

Approach to Analysis

The simplest method to deal with the analysis of unbalanced conditions is the use of the concept of positive and negative sequence components. However, the presence of SCRs prevent its ready application due to the presence of non-sinusoidal currents. The analysis of induction machines with symmetrically triggered thyristor control has been thoroughly investigated in a previous paper [10] using the state variable approach to calculate the performance of the the system in a $d-q$ axis formulation. In particular, the three phase and half cycle symmetry of the problem permits a closed form calculation of the steady state initial condition. A closed form solution over only 60° is needed to completely define the solution. Unfortunately, the possibility of an unbalanced power supply as well as unsymmetrical triggering does not permit the previously obtained solution to be readily extended to the problem of phase unbalance. Securing of the desired solution indeed becomes a formidable task. The required initial condition must now be found by iteration since the most of the benefits of a symmetrical operating condition does not exist. The only symmetrical property that can now be utilized is half wave symmetry in which the waveform is repeated for each half period. The system matrix which represents each of the possible connections of the machine must be derived separately.

A simplified circuit diagram of a voltage controller is given in Fig. 1. The system consists of three pairs of identical thyristors, connected back to back in series with the phases of a three phase wye connected induction machine. Figure 2 shows a typical line current which serves to define the electrical angles which are used in the analysis. In general, stator voltage control is accomplished by adjusting the hold off angle γ in which the thyristor is kept open during a current reversal in a given phase. With unsymmetrical firing distinctly different values of γ may occur in each phase. The time between open circuit intervals are identified by the conduction angles β. Over a complete half cycle three distinct values of γ and β can be identified associated with each of the three phases.

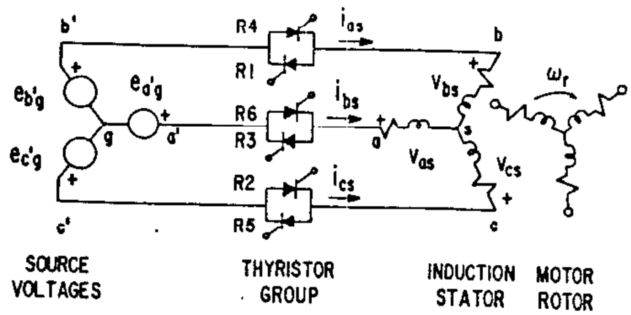

Fig. 1 Voltage Controller Circuit Configuration

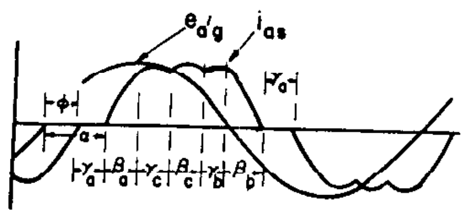

Fig. 2 Typical Motor Line Current Waveform Identifying Hold Off and Conduction Angles γ and β.

It can be noted that there exist five possible states which can occur during a entire period.

1) All three phases connected to the power supply; ias, ibs, ics are non zero.
2) Phase as disconnected from the supply ; ias = 0.
3) Phase bs disconnected from the supply ; ibs = 0.
4) Phase cs disconnected from the supply ; ics = 0.
5) All three phases disconnected from the supply.

Of these five states, only the first four need be discussed. The last state occurs only when the system enters the mode in which either one or none of the phases are connected to the power supply [10]. Since operation in this mode results in very low average torque, this state is not of interest here.

With proper attention to the constraints which exist for each case, the motor equations which define the open circuit condition for each of the three phases can be obtained. The approach utilized in Ref. 10 is used which, in turn, is an extension of Ref. 11. The system equations for each of the four circuit states needed to define the operating condition is summarized in Appendix 1. The overall solution can now be obtained by proper attention to the symmetry available (half wave symmetry). It is shown in Appendix 2 that a solution in state variable form over a half cycle leads to the equation

$$[e^{\beta_a \mathbf{A}} e^{\gamma_a \mathbf{B}_a} e^{\beta_c \mathbf{A}} e^{\gamma_c \mathbf{B}_c} e^{\beta_b \mathbf{A}} e^{\gamma_b \mathbf{B}_b} + \mathbf{I}]\, \mathbf{x}(\varphi) = 0$$

The matrices $\mathbf{A}$, $\mathbf{B}_a$, $\mathbf{B}_b$ and $\mathbf{B}_c$ are the system state matrices for the system states 1-4 described above. The quantity $\mathbf{x}(\varphi)$ is the system state variable vector at the initial condition $\omega_e t = \varphi$. This equation is used to calculate the value of φ and the initial value of the stator current, rotor current and source voltages. Since the values of hold off angles $\gamma_a, \gamma_b, \gamma_c$ are assumed known, the values of β_b and β_c are then the remaining unknown quantities since the third angle β_b can be found from the half wave symmetry. That is

$$\beta_a = \pi - \gamma_a - \gamma_b - \gamma_c - \beta_b - \beta_c$$

Solutions for β_b and β_c must be iterated to fulfill the zero crossing current constraint. Fortunately it turns out that the zero crossing currents are related almost linearly with β_b and β_c so that an iterative solution converges rapidly.

Phase Balancing Strategy

Since the SCR voltage controller of Fig. 1 is essentially used to adjust the fundamental component of the voltages at the terminals of the load, the SCR controller itself can be viewed as an equivalent variable impedance. Since the SCR does not absorb or produce energy it can be shown that the fundamental component of voltage across the SCR is in quadrature with the fundamental component of current. Hence, the SCR controller is, in effect, a variable reactor. Figure 3 illustrates a typical unbalance case in which a phase voltage V_a is larger and phase leading with respect to its desired position. The terminal voltage V_a can be viewed as being comprised of a positive sequence component V_a^+ and a negative sequence component V_a^- wherein the positive sequence component is effectively the desired value of the phase voltage. Phase control of the pair of SCRs in series with this phase serve to reduce the voltage drop across the load. The load voltage phasor locus follows a line which is perpendicular to the phase current phasor (assuming that the current phasor remains unchanged during the adjustment). The unbalance condition generally be can be reduced to a certain minimum (denoted as V_a') before the negative sequence component again begins to increase (as, for example, V_a'').

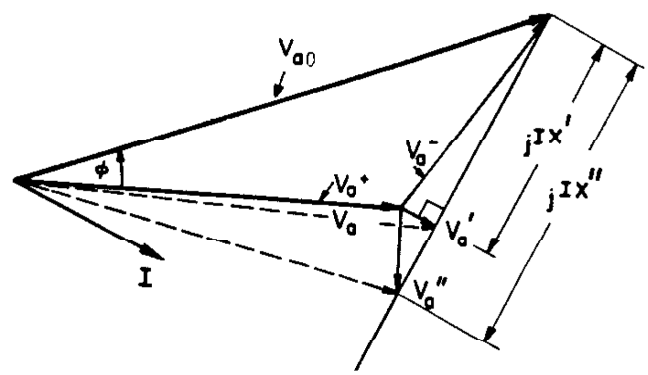

Fig. 3 Phasor Diagram Illustrating Principle of Phase Balancing.

A pictorial representation of the overall phase balancing strategy proposed in this paper is shown in Fig. 4. While the proposed overall strategy can be extended to the case of a general phase unbalance, it is apparent that a comprehensive treatment of such a general case is not feasible in a paper with space constraints. For simplicity, only two different phase unbalance conditions are considered and are shown in Fig. 3. These conditions correspond to (a) a single phase unbalance in which phase *as* is a different amplitude and retarded in phase relative to phases *bs* and *cs* and (b) a double phase unbalance in which phases *bs* and *cs* are unsymmetric relative to phase *as*. In each case the unsymmetric phasor is assumed to be oriented in a phase lag position compared to its normal position. Hence, the cases studied correspond to practical single phase or two phase resistive or inductive unbalanced loads. Note that in case (b) the voltage phasors V_{bs} and V_{cs} are assumed to be equal (but different from V_{as}) and phase displaced by an equal amount φ.

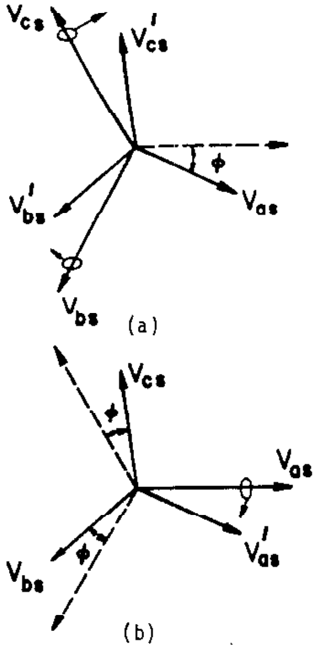

Fig. 4 Phasor Diagrams Showing Strategy for (a) Single Phase Unbalance and (b) Double Phase Unbalance.

The strategy to correct a single phase unbalance, Fig. 4(a), is clearly to retard the firing of the SCRs connected in series with phases *bs* and *cs*. If the hold off angle of the SCRs in each of these phases are controlled to be equal, the two phases voltages will be displaced from their normal position by nearly the same angle. It is apparent that in the optimization procedure the SCRs in phases *bs* and *cs* must be retarded in phase by a hold off angle γ approximately equal to φ. This observation provides a good starting point to begin searching for the most desirable value of γ. The final location of the voltage phasors of *bs* and *cs* are given by V_{bs}' and V_{cs}'.

In case (b) two phases are unbalanced relative to the third. In this case the unbalance can be corrected by retarding the third phase to bring this phase into a more symmetrical relationship with respect to the two unbalanced components. The final location of the phase *as* voltage vector is denoted by V_{as}'. It should be apparent that if a substantial improvement in phase balance can be achieved in these two cases that similar improvements can be realized in a general unsymmetrical condition. However, in this case the control angles of all three phases must be adjusted independently if a minimum unbalanced condition is to be realized.

It should be noted that due to the active load, a phase retard as shown in Fig. 4(a) or 4(b) does not only affect the negative sequence component as idealized in Fig. 3 but also has a smaller but important effect on the positive sequence. Hence, if the output power or torque is to remain unchanged, the motor is required to operate with a somewhat greater slip. Although the negative sequence losses may be reduced by the phase retard, positive sequence losses may increase at a faster rate resulting in an overall efficiency loss rather than efficiency gain. This problem can be avoided by supplying a somewhat greater than rated voltage to the motor. By proper choice of the turns ratio and by keeping the SCRs continuously active the voltage can be maintained as its rated value over the entire expected range of phase unbalance. Although this strategy increases somewhat the losses under balanced conditions, for practical cases the amount of control angle γ is only a few degrees. Hence, the added losses under balanced conditions are expected to be very small.

Results of Optimization Study

Although the hold off angle γ can be expected to vary somewhat in proportion to the phase angle φ, the optimum value can only be obtained by means of iteration. Figure 5 shows how the negative sequence current varies with the control angle γ for a particular case of double phase unbalance. Note that approximately a 4 times reduction in negative sequence current is possible resulting in a reduction of negative sequence copper losses by about a factor of 16.

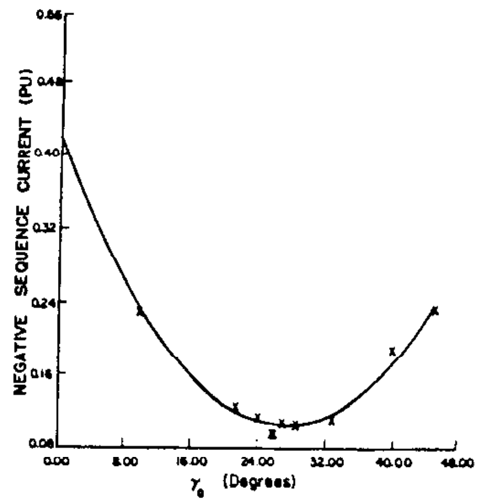

Fig. 5 Negative Sequence Current as a Function of Hold Off Angle γ_a for Double Phase Unbalance, $|V_{bs}| = |V_{cs}| = 0.92$ pu, $\varphi = -10°$.

Although very large improvements can be made in reducing the negative sequence component, this reduction is somewhat offset by the increased harmonic losses. Figures 6, 7 and 8 show the negative sequence current, losses in each phase and total losses for the same case of a double phase unbalance when the magnitude of the two unbalanced phases vary from 0.92 to 1.06 of rated voltage. The parameters used for this computation are given in Appendix 3. Specifically, the motor selected was a 220 Volt machine

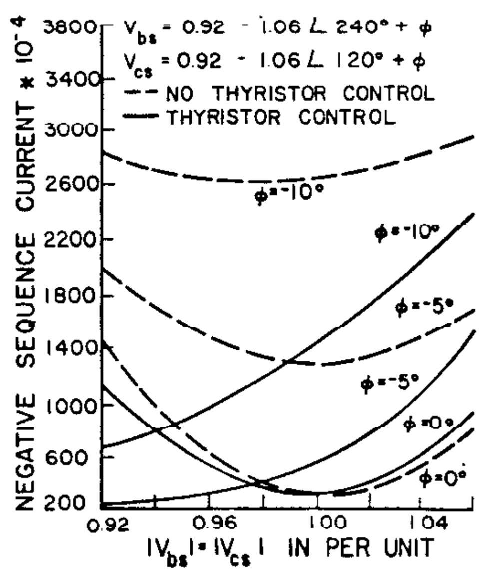

Fig. 6 Negative Sequence Current as a Function of the Degree of Voltage Unbalance in Per Unit with Double Phase Unbalance.

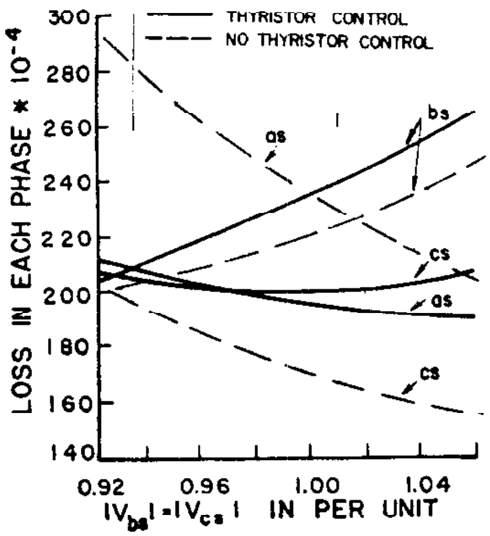

Fig. 7 Losses in Individual Motor Phases as a Function of the Degree of Voltage Unbalance for Double Phase Unbalance, $\varphi = -5°$.

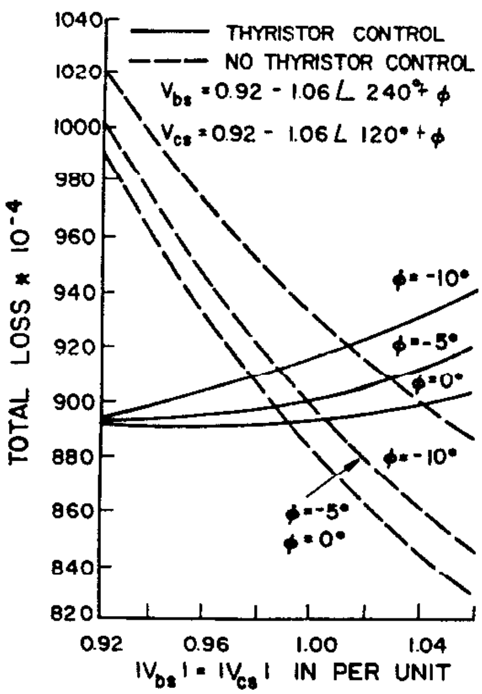

Fig. 8 Total Motor Losses as a Function of the Degree of Voltage Unbalance for Double Phase Unbalance.

and the transformer secondary voltage was assumed to be 240 V. Hence, the minimum unbalance voltage amplitude that can be corrected without reducing the fundamental component is 0.92 pu. Note that substantial in negative sequence current is possible when the per unit unbalance voltage is less than one, phase retarding will not correct situations in which the unbalance voltage is greater than unity. Fortunately such cases arise with unbalanced capacitive loads, an unlikely occurence. Note also that correction of unbalance is more attractive for moderate phase unbalances ($\varphi = -5°$) than when the phase unbalance is zero or very large (significantly worse than - 10°).

Figures 9, 10 and 11 show corresponding data for the case of the single phase unbalance. In this case, it should be noted that a somewhat larger transformer secondary voltage is required in order to achieve the phase correction required. While not necessary for the case of $\varphi = 0°$ or $\varphi = -5°$, the transformer secondary voltage was assumed to be 250 V for the plots of $\varphi = -10°$ in Figs. 9-11. The increased secondary voltage decreases somewhat the improvement that can be achieved for $\varphi = -10°$ since the harmonic losses increases. In a practical application it appears that the transformer turns ratio must be carefully chosen to be consistent with the maximum expected phase unbalance.

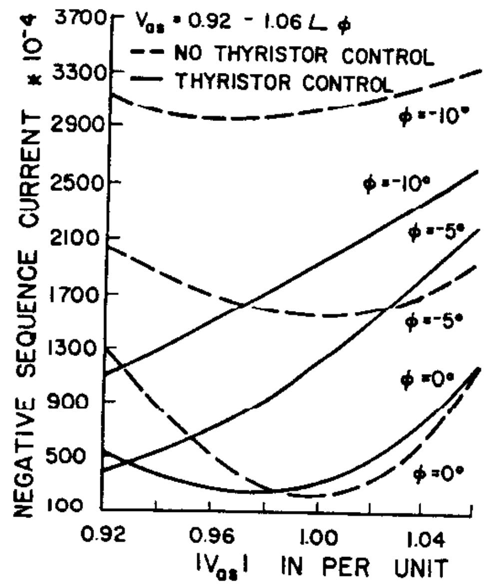

Fig. 9 Negative Sequence Current as a Function of Unbalanced Voltage for a Single Phase Unbalance, $\varphi = -5°$.

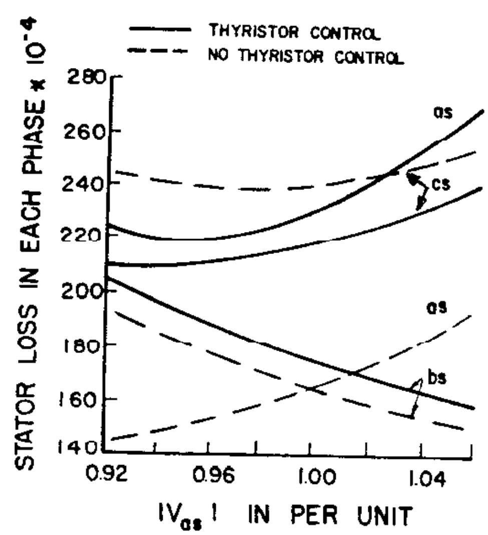

Fig. 10 Losses in Individual Motor Phases as a Function of Unbalanced Voltage Amplitude for Single Phase Unbalance.

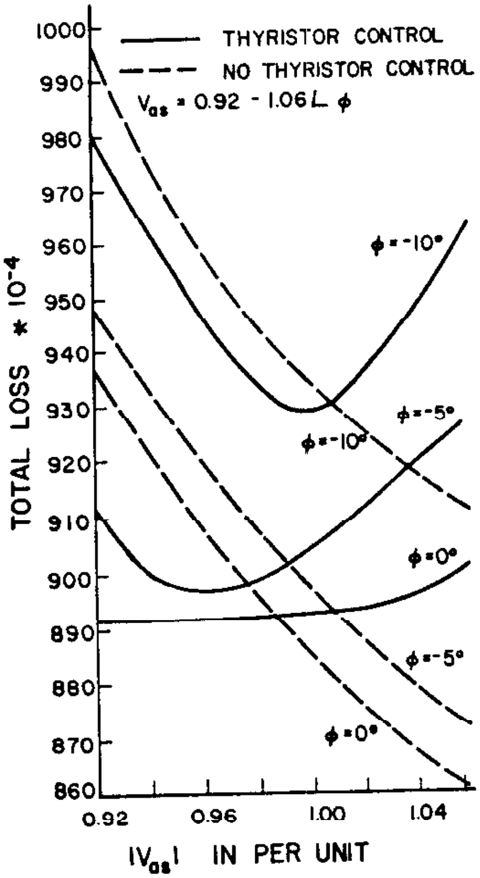

Fig. 11 Total Motor Losses as a Function of the Degree of Voltage Unbalance for Single Phase Unbalance.

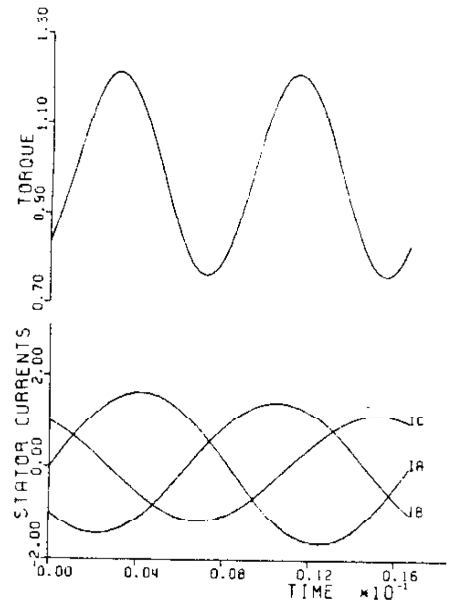

Fig. 12 Electromagnetic Torque and Motor Line Currents at Rated Load versus Time, $|V_{as}|=1.0$, $|V_{bs}|=|V_{cs}|=0.95$, φ = -10°. Hold Off Angles of All Three Phases Set to Zero. Supply Voltage Equal to 220 V.

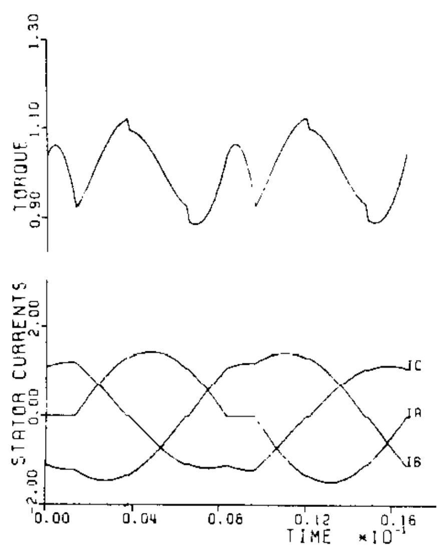

Fig. 13 Electromagnetic Torque and Motor Line Currents versus Time for Same Condition as Fig. 12 Using Phase Balancing. Supply Voltage Set to 240 V, γ_a = 28.7°, γ_b = 2.7°, γ_c = 2.7°.

Although Figs. 6-11 are sufficient to demonstrate the feasibility of asymmetric control, it is useful to show typical results in the time domain for a particular unbalance condition to help visualize the nature of the compensation technique. Figure 12 shows a time domain solution for the motor stator currents and electromagnetic torque for the particular case of double unbalance where the per unit unbalance voltage amplitude is 0.95 and the unbalance phase shift φ = -10°. The supply voltage is assumed to be equal to the motor rating. That is one per unit voltage corresponds to 220 V. When the angle γ is set equal to zero (no control) the resulting large unbalance currents and pulsating torque due to the interaction of the positive and negative sequence current components is readily observed. In Fig. 13 the hold off angle of phase *as* has been adjusted to minimize the negative sequence current while the hold off angles of phases *bs* and *cs* have been set to achieve rated positive sequence voltage. A significant reduction in current unbalance and pulsating torque is readily apparent.

Figures 14 and 15 show a similar pair of plots for the single phase unbalance condition. In this case the hold off angles of phases *bs* and *cs* are adjusted equally to 29° while the hold off angle of phase *as* is set for rated positive sequence voltage (γ_a = 0°). Again, a substantial reduction in current unbalance and pulsating is immediately apparent.

It should be emphasized that in each case the hold off angles for phases *bs* and *cs* were adjusted in equal fashion while the hold off angle for phase *as* was varied independently. In this manner the dual constraints of achieving rated positive sequence voltage and minimum negative sequence current could be conveniently obtained. No attempt was made to adjust all three angles independently since the added degree of freedom would have made the optimization procedure much more difficult. It can be safely stated that the "optimum" obtained by the iteration procedure merely forms an upper bound for the loss reduction that can be obtained and even better solutions may be possible if the three control angles are varied independently. A detailed study exploring this possibility could be the subject of future work.

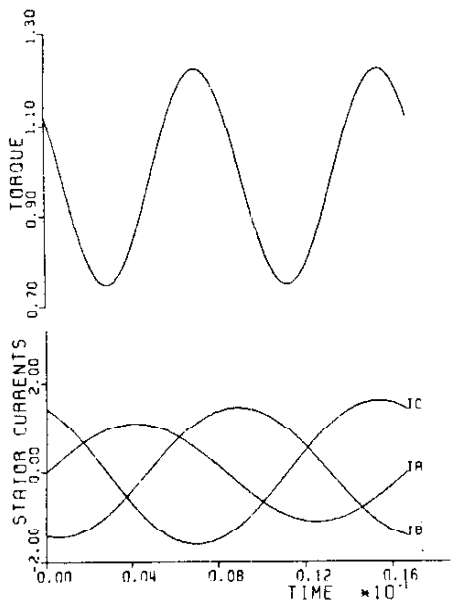

Fig. 14 Electromagnetic Torque and Motor Line Currents at Rated Load versus Time, $|V_{as}|=0.92$, $|V_{bs}|=|V_{cs}|=1.0$, $\varphi = -10°$. Hold Off Angles of All Three Phases Set to Zero. Supply Voltage Equal to 220 V.

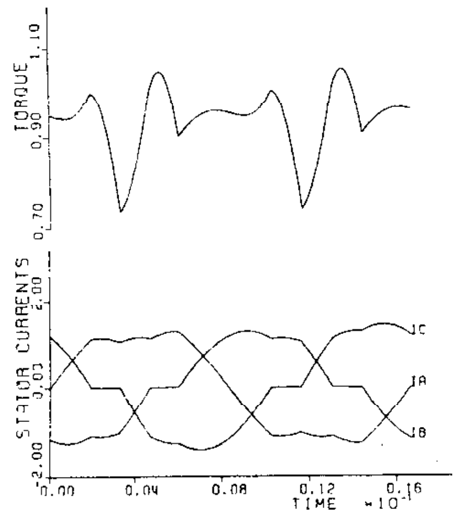

Fig. 15 Electromagnetic Torque and Motor Line Currents versus Time for Same Condition as Fig. 14 Using Phase Balancing. Supply Voltage Set to 240 V, $\gamma_a = 0°$, $\gamma_b = 29°$, $\gamma_c = 29°$.

Conclusion

This paper has proposed a new technique for dynamic balancing the phase voltages of an induction motor when subjected to unbalanced supply. It is shown that by proper application of components which may already be present in a system, i.e. isolation transformer and static starter, the negative sequence current can reduced by a factor of five or even more. While the technique introduces added harmonic losses due to the presence of solid state switches, it is shown than an overall improvement can be readily obtained.

Acknowledgments

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Appendix 1
System Equations

Induction Machine Equations

The equations for an three phase, three wire induction machine expressed in a stationary $d-q$ axis are:

$$\begin{bmatrix} v_{qs}^s \\ v_{ds}^s \\ v_{qr}^s \\ v_{dr}^s \end{bmatrix} = \begin{bmatrix} r_s + (p/\omega_b)X_s & 0 & (p/\omega_b)X_m & 0 \\ 0 & r_s + (p/\omega_b)X_s & 0 & (p/\omega_b)X_m \\ (p/\omega_b)X_m & -(\omega_r/\omega_b)X_m & r_r + (p/\omega_b)X_r & -(\omega_r/\omega_b)X_r \\ (\omega_r/\omega_b)X_m & (p/\omega_b)X_m & (\frac{\omega_r}{\omega_b})X_r & r_r + (p/\omega_b)X_r \end{bmatrix} \begin{bmatrix} i_{qs}^s \\ i_{ds}^s \\ i_{qr}^s \\ i_{dr}^s \end{bmatrix} \quad \text{(A1.1)}$$

The superscript s denotes the stationary reference frame. Since this frame is used throughout this paper, the superscript s will be dropped here on for simplicity.

The above matrix equation can be expressed in the simpler vector matrix form:

$$\mathbf{v} = \mathbf{X}\,(\frac{p}{\omega_b})\,\mathbf{i} + \mathbf{R}\,\mathbf{i} \quad \text{(A1.2)}$$

The stator voltages in $d-q$ axis are related to the phase voltages by the equations:

$$v_{qs} = v_{as} \quad \text{(A1.3)}$$

$$v_{ds} = \frac{1}{\sqrt{3}}(v_{cs} - v_{bs}) \quad \text{(A1.4)}$$

Similarly the $d-q$ currents are expressed in terms of the phase currents by

$$i_{qs} = i_{as} \tag{A1.5}$$

$$i_{ds} = \frac{1}{\sqrt{3}}(i_{cs} - i_{bs}) \tag{A1.6}$$

The per unit electromagnetic torque can be expressed in $d-q$ axis variables as

$$T_e = X_m (i_{qs} i_{dr} - i_{ds} i_{qr}) \tag{A1.7}$$

Supply Voltage Equations for Unbalanced Condition

Since a three wire system is assumed the supply voltages are conveniently expressed as line voltages.

$$e_{ab} = E_{ab} \sin(\omega_e t + \varphi_{ab}) \tag{A1.8}$$

$$e_{bc} = E_{bc} \sin(\omega_e t - 120^\circ + \varphi_{bc}) \tag{A1.9}$$

$$e_{ca} = E_{ca} \sin(\omega_e t + 120^\circ + \varphi_{ca}) \tag{A1.10}$$

where

$$e_{ab} = e_{ag} - e_{bg} \tag{A1.11}$$

$$e_{bc} = e_{bg} - e_{cg} \tag{A1.12}$$

$$e_{ca} = e_{cg} - e_{ag} \tag{A1.13}$$

and wherein

$$e_{ag} = E_{ag} \sin(\omega_e t + \varphi_a) \tag{A1.14}$$

$$e_{bg} = E_{bg} \sin(\omega_e t - 120^\circ + \varphi_b) \tag{A1.15}$$

$$e_{cg} = E_{cg} \sin(\omega_e t + 120^\circ + \varphi_c) \tag{A1.16}$$

When the machine consists of a three wire system the sum of the three motor line to neutral voltages must sum to zero [10]. It can be shown that the motor voltages can then be expressed in terms of the line source voltages as

$$\begin{bmatrix} v_{as} \\ v_{bs} \\ v_{cs} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} e_{ab} \\ e_{bc} \\ e_{ca} \end{bmatrix} \tag{A1.17}$$

From Eqs. A1.3 and A1.4

$$v_{qs} = \frac{V_{ab}}{3}(\sin\varphi_{ab} \cos\omega_e t + \cos\varphi_{ab} \sin\omega_e t) + \frac{V_{ca}}{3}[\sin(\varphi_{ca} - 120^\circ)\cos\omega_e t + \cos(\varphi_{ca} - 120^\circ)\sin\omega_e t] \tag{A1.18}$$

$$v_{ds} = \frac{V_{bc}}{\sqrt{3}}[\sin(\varphi_{bc} + 120^\circ)\cos\omega_e t + \cos(\varphi_{bc} + 120^\circ)\sin\omega_e t] \tag{A1.19}$$

or in a simple matrix form:

$$\mathbf{v} = \mathbf{C}\,\mathbf{e} \tag{A1.20}$$

From this result the matrix $\mathbf{C}$ can be derived,

$$\mathbf{C} = \begin{bmatrix} \frac{1}{3}[V_{ab}\cos\varphi_{ab} + V_{ca}\cos(\varphi_{ca} + 120^\circ)] & \frac{1}{3}[V_{ab}\sin\varphi_{ab} + V_{ca}\sin(\varphi_{ca} + 120^\circ) \\ \frac{1}{\sqrt{3}}V_{bc}\cos(\varphi_{bc} - 120^\circ) & \frac{1}{\sqrt{3}}V_{bc}\sin(\varphi_{bc} - 120^\circ) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A1.21}$$

where

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \sin\omega_e t \\ \cos\omega_e t \end{bmatrix} \tag{A1.22}$$

The required sine and cosine functions of Eq. A1.22 can be readily derived an auxiliary set of differential equations, namely

$$\frac{p}{\omega_e} e_1 = e_2 \tag{A1.23}$$

$$\frac{p}{\omega_e} e_2 = -e_1 \tag{A1.24}$$

where the initial conditions $e_1 = 0$ and $e_2 = 1$ are assumed. In matrix form

$$\frac{p}{\omega_e}\mathbf{e} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{e} \tag{A1.25}$$

System Matrix Equation for Full Conduction

The differential equation for the case where three phase are connected to the power supply may now be expressed in matrix notation as

$$\frac{p}{\omega_b}\mathbf{i} = -\mathbf{X}^{-1}\mathbf{R}\mathbf{i} + \mathbf{X}^{-1}\mathbf{C}\mathbf{e} \tag{A1.26}$$

$$\frac{p}{\omega_e}\mathbf{e} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{e} \tag{A1.27}$$

These two matrices can be combined as one matrix to form:

$$\frac{p}{\omega_e}\begin{bmatrix} \mathbf{i} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} -\mathbf{X}^{-1}\mathbf{R} & \mathbf{X}^{-1}\mathbf{C} \\ 0 & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{e} \end{bmatrix} \tag{A1.28}$$

or, in state variable matrix notation:

$$\frac{p}{\omega_e}\mathbf{x} = \mathbf{A}\mathbf{x} \tag{A1.29}$$

The contents of matrices $\mathbf{X}$ and $\mathbf{R}$ are:

$$\mathbf{X} = \begin{bmatrix} X_s & 0 & X_m & 0 \\ 0 & X_s & 0 & X_m \\ X_m & 0 & X_r & 0 \\ 0 & X_m & 0 & X_r \end{bmatrix} \tag{A1.30}$$

$$\mathbf{R} = \begin{bmatrix} r_s & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 \\ 0 & -(\frac{\omega_r}{\omega_b})X_m & r_r & -(\frac{\omega_r}{\omega_b})X_r \\ (\frac{\omega_r}{\omega_b})X_m & 0 & (\frac{\omega_r}{\omega_b})X_r & r_r \end{bmatrix} \tag{A1.31}$$

Phase *as* Disconnected from the Supply

When phase *as* is disconnected from the power supply only two phases excite the windings. The current in phase *as* equal zero so that $i_{qs} = 0$. The resulting open circuit voltage in phase *as* may be expressed in the $d-q$ axes as:

$$v_{qs} = \frac{p}{\omega_e} X_m i_{qr} \tag{A1.32}$$

Since the *bs* and *cs* phases remain connected to the induction machine, the voltage along the *ds* axis is given by:

$$v_{ds} = \frac{1}{\sqrt{3}}(e_{cg} - e_{bg}) \tag{A1.33}$$

The total matrix equation is essentially the same as the full conduction mode except that the matrices $\mathbf{X}$ and $\mathbf{C}$ are simply changed to $\mathbf{X}_a$ and $\mathbf{C}_a$, wherein

$$\mathbf{X}_a = \begin{bmatrix} X_s & 0 & 0 & 0 \\ 0 & X_s & 0 & X_m \\ X_m & 0 & X_r & 0 \\ 0 & X_m & 0 & X_r \end{bmatrix} \tag{A1.34}$$

$$\mathbf{C}_a = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} V_{bc}\cos(\varphi_{bc}-120°) & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} V_{bc}\sin(\varphi_{bc}-120°) & 0 & 0 & 0 \end{bmatrix}^T \qquad (A1.35)$$

The total matrix expression for this mode becomes:

$$\frac{p}{\omega_e}\begin{bmatrix}\mathbf{i}\\ \mathbf{e}\end{bmatrix} = \begin{bmatrix} -\mathbf{X}_a^{-1}\mathbf{R} & \mathbf{X}_a^{-1}\mathbf{C}_a \\ [0] & \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\end{bmatrix}\begin{bmatrix}\mathbf{i}\\ \mathbf{e}\end{bmatrix} \qquad (A1.36)$$

or in state variable matrix form:

$$\frac{p}{\omega_e}\mathbf{x} = \mathbf{B}_a\,\mathbf{x} \qquad (A1.37)$$

Phase cs Disconnected from the Power Supply

The equations in this state are slightly different from the previous derivation when phase a is disconnected from the supply. Instead of using v_{qs} and v_{ds} in the main equation, the stator voltage is denoted by the remaining connected voltage supply. The matrix equation for the induction machine is expressed as:

$$\begin{bmatrix} V_{ab} \\ V_{ab} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2[r_s + (\frac{p}{\omega_b})X_s] & 0 & \frac{3}{2}(\frac{p}{\omega_b})X_m & \frac{\sqrt{3}}{2}(\frac{p}{\omega_b})X_m \\ 0 & -2\sqrt{3}[r_s + (\frac{p}{\omega_b})X_s] & \frac{3}{2}(\frac{p}{\omega_b})X_m & \frac{\sqrt{3}}{2}(\frac{p}{\omega_b})X_m \\ (\frac{p}{\omega_b})X_m & -\frac{\omega_r}{\omega_b}X_m & r_r + (\frac{p}{\omega_b})X_r & -(\frac{\omega_r}{\omega_b})X_r \\ (\frac{\omega_r}{\omega_b})X_m & (\frac{p}{\omega_b})X_m & (\frac{\omega_r}{\omega_b})X_r & r_r + (\frac{p}{\omega_b})X_r \end{bmatrix} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qr} \\ i_{dr} \end{bmatrix} \qquad (A1.38)$$

From the above matrix representation, matrix $\mathbf{R}_c$, matrix $\mathbf{X}_c$ and matrix $\mathbf{C}_c$ can be derived and may be written as follows:

$$\mathbf{R}_c = \begin{bmatrix} 2r_s & 0 & 0 & 0 \\ 0 & 2\sqrt{3}r_s & 0 & 0 \\ 0 & -\frac{\omega_r}{\omega_b}X_m & r_r & -\frac{\omega_r}{\omega_b}X_r \\ \frac{\omega_r}{\omega_b}X_m & 0 & \frac{\omega_r}{\omega_b}X_r & r_r \end{bmatrix} \qquad (A1.39)$$

$$\mathbf{X}_c = \begin{bmatrix} 2X_s & 0 & \frac{3}{2}X_m & \frac{\sqrt{3}}{2}X_m \\ 0 & 2\sqrt{3}X_s & \frac{3}{2}X_m & \frac{\sqrt{3}}{2}X_m \\ X_m & 0 & X_r & 0 \\ 0 & X_m & 0 & X_r \end{bmatrix} \qquad (A1.40)$$

$$\mathbf{C}_c = \begin{bmatrix} V_{ab}\cos\varphi_{ab} & V_{ab}\sin\varphi_{ab} \\ V_{ab}\cos\varphi_{ab} & V_{ab}\sin\varphi_{ab} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (A1.41)$$

Phase bs Disconnected from the Power Supply

The derivation for this state is very similar to the derivation for phase cs disconnected from the supply. The stator voltage used in the main equation is also the remaining connected line voltage. The following matrix represents the main matrix of induction machine, matrix $\mathbf{R}_b$, matrix $\mathbf{X}_b$, matrix $\mathbf{C}_b$ and the system matrix.

$$\begin{bmatrix} V_{ac} \\ V_{ac} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2[r_s + (\frac{p}{\omega_b})X_s] & 0 & \frac{3}{2}(\frac{p}{\omega_b})X_m & -\frac{\sqrt{3}}{2}(\frac{p}{\omega_b})X_m \\ 0 & -2\sqrt{3}[r_s + (\frac{p}{\omega_b})X_s] & \frac{3}{2}(\frac{p}{\omega_b})X_m & -\frac{\sqrt{3}}{2}(\frac{p}{\omega_b})X_m \\ (\frac{p}{\omega_b})X_m & -\frac{\omega_r}{\omega_b}X_m & r_r + (\frac{p}{\omega_b})X_r & -\frac{\omega_r}{\omega_b}X_r \\ \frac{\omega_r}{\omega_b}X_m & (\frac{p}{\omega_b})X_m & \frac{\omega_r}{\omega_b}X_r & r_r + (\frac{p}{\omega_b})X_r \end{bmatrix} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{qr} \\ i_{dr} \end{bmatrix} \qquad (A1.42)$$

From the above matrix representation, $\mathbf{R}_b$, $\mathbf{X}_b$ and $\mathbf{C}_b$ can be derived and may be written as follows:

$$\mathbf{R}_b = \begin{bmatrix} 2r_s & 0 & 0 & 0 \\ 0 & -2\sqrt{3}r_s & 0 & 0 \\ 0 & -\frac{\omega_r}{\omega_b}X_m & r_r & -\frac{\omega_r}{\omega_b}X_r \\ \frac{\omega_r}{\omega_b}X_m & 0 & \frac{\omega_r}{\omega_b}X_r & r_r \end{bmatrix} \qquad (A1.43)$$

$$\mathbf{X}_b = \begin{bmatrix} 2X_s & 0 & \frac{3}{2}X_m & -\frac{\sqrt{3}}{2}X_m \\ 0 & -2\sqrt{3}X_s & \frac{3}{2}X_m & -\frac{\sqrt{3}}{2}X_m \\ X_m & 0 & X_r & 0 \\ 0 & X_m & 0 & X_r \end{bmatrix} \qquad (A1.44)$$

$$\mathbf{C}_b = \begin{bmatrix} V_{ca}\cos(\varphi_{ca}+120°) & V_{ca}\sin(\varphi_{ca}+120°) \\ V_{ca}\cos(\varphi_{ca}+120°) & V_{ca}\sin(\varphi_{ca}+120°) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (A1.45)$$

Appendix 2
Initial Condition for Unbalanced Operation

The initial condition for an unbalanced supply cannot be found in a closed form as was the case for the balanced condition. However, the initial condition can still be found by utilizing half wave symmetry. The derivation of the initial condition is based on the assumption that the first guess of the hold off angle affects the final value for each full conduction interval (state 1) which should always end at a zero crossing of one of the currents. Based on this assumption, the actual zero crossing point is calculated and the result is used as a feedback to iterate the new conduction angle and to recalculate the initial condition angle. The iteration is repeated until the firing angles converge to the point where the total period is close to π radians. The waveform of a typical current in phase *as* for the entire period is given in Fig. 2.

Using the same derivation as for the balanced condition and choosing the same starting point $\omega_e t = \varphi$, the solution during the interval when phase *as* is disconnected from the supply is [9]

$$\mathbf{x}(\omega_e t) = e^{(\omega_e t-\varphi)\mathbf{B}_a}\,\mathbf{x}(\varphi) \qquad \varphi<\omega_e t<\varphi+\gamma_a \tag{A2.1}$$

Hence, when $\omega_e t = \varphi+\gamma_a$:

$$\mathbf{x}(\varphi + \gamma_a) = e^{\gamma_a\mathbf{B}_a}\,\mathbf{x}(\varphi) \tag{A2.2}$$

The solution during the full conduction interval which occurs after phase *as* is interrupted is

$$\mathbf{x}(\omega_e t) = e^{(\omega_e t-\gamma_a)\mathbf{A}}\mathbf{x}(\gamma_a) \qquad \varphi+\gamma_a<\omega_e t<\varphi+\gamma_a+\beta_a \tag{A2.3}$$

At $\omega_e t = \varphi+\gamma_a+\beta_a$,

$$\mathbf{x}(\gamma_a + \beta_a) = e^{\beta_a\mathbf{A}}\mathbf{x}(\gamma_a) \tag{A2.4}$$

The solution during the time phase *cs* is turned off can be written,

$$\mathbf{x}(\omega_e t) = e^{(\omega_e t-\gamma_a-\beta_a)\mathbf{B}_c}\mathbf{x}(\gamma_a+\beta_a) \tag{A2.5}$$

$$\varphi+\gamma_a+\beta_a<\omega_e t<\varphi+\gamma_a+\beta_a+\gamma_c$$

and, at the instant $\omega_e t=\varphi+\gamma_a+\beta_a+\gamma_c$,

$$\mathbf{x}(\varphi+\gamma_a+\beta_a+\gamma_c) = e^{\gamma_c\mathbf{B}_c}\mathbf{x}(\varphi+\gamma_a+\beta_a) \tag{A2.6}$$

The solution can be continued in this manner during the remainder of one-half cycle. The resulting equations are:

$$\mathbf{x}(\varphi+\gamma_a+\beta_a+\gamma_c+\beta_c) = e^{\gamma_c\mathbf{A}}\mathbf{x}(\varphi+\gamma_a+\beta_a+\gamma_c) \tag{A2.7}$$

$$\mathbf{x}(\gamma_a+\beta_a+\gamma_c+\beta_c) = e^{\gamma_b\mathbf{B}_b}\mathbf{x}(\gamma_a+\beta_a+\gamma_c+\beta_c) \tag{A2.8}$$

$$\mathbf{x}(\pi+\varphi) = e^{\beta_b\mathbf{A}}\mathbf{x}(\pi-\beta_b) \tag{A2.9}$$

Combining Eqs. A2.2, A2.4 and A2.6-A2.9 the total solution for a half cycle becomes

$$\mathbf{x}(\pi+\varphi) = e^{\beta_a\mathbf{A}}e^{\gamma_a\mathbf{B}_b}e^{\beta_c\mathbf{A}}e^{\gamma_c\mathbf{B}_c}e^{\beta_b\mathbf{A}}e^{\gamma_b\mathbf{B}_b}\mathbf{x}(\varphi) \tag{A2.10}$$

and from half wave symmetry:

$$\mathbf{x}(\pi+\varphi) = -\mathbf{x}(\varphi) \tag{A2.11}$$

The two equations can be combined together to get the final expression:

$$[e^{\beta_a\mathbf{A}}e^{\gamma_a\mathbf{B}_a}e^{\beta_c\mathbf{A}}e^{\gamma_c\mathbf{B}_c}e^{\beta_b\mathbf{A}}e^{\gamma_b\mathbf{B}_b} + \mathbf{I}]\,\mathbf{x}(\varphi) = 0 \tag{A2.12}$$

This equation is used to calculate the value of φ and the initial value of the stator, rotor current and source voltages. The solution for the initial phase angle is obtained in the same manner as for the balanced condition [9]. The values of hold off angles $\gamma_a, \gamma_b, \gamma_c$ are known and the value of $\beta_a = \pi-\gamma_a-\gamma_b-\gamma_c-\beta_b-\beta_c$ are then the remaining unknown values are β_b and β_c have to be iterated to fulfill the zero crossing current constraint. Fortunately, the zero crossing currents are related almost linearly with β_b and β_c. Hence, the iterations converge rapidly.

Appendix 3
Induction Motor Data

Nameplate Data	Baldor Mfg. Co.	3 Phase
	220 Volts	20 Amps
	1725 RPM	60 Hz
	1.15 S.F.	55°C
Base Quantities	P_B = 5595 W	V_B = 127.02 V
	ω_b = 377 rad/sec	I_b = 14.68 A
	Z_b = 8.65 Ω	T_b = 29.68 Nt-m
Parameter Values	r_s = 0.023 pu	r_r' = 0.014 pu
	X_{ls} = 0.1045 pu	X_{lr}' = 0.1045 pu
	X_m = 1.4686 pu	