

AN IMPROVED WEIGHTED TOTAL HARMONIC DISTORTION INDEX FOR INDUCTION MOTOR DRIVES

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Abstract: *The weighted total harmonic distortion (WTHD) is a commonly used expression to assess the quality of pulse width modulated (PWM) inverter waveforms. The WTHD weights the voltage harmonics inversely with its frequency. While this is adequate for some inductor type loads, the commonly employed induction motor load has important effects resulting from eddy currents in the rotor bars not incorporated in the WTHD. In this paper a new distortion index, the IMWTHD, is proposed to more accurately evaluate modulated inverter waveform quality for squirrel-cage induction motor loads.*

Key words: Harmonic Distortion, Total Harmonic Distortion, Weighted Harmonic Distortion, THD, WTHD, Harmonics, Pulse Width Modulation.

1. Introduction

The proliferation of converter modulation algorithms that have appeared over the years has created confusion as to the effectiveness of one method over another for comparing various algorithms. One method of comparing the effectiveness of modulation processes is by comparing the unwanted components, i.e. the “distortion”, in the output current waveform relative to that of an ideal sine wave.

The most commonly used distortion index is the weighted total harmonic distortion (WTHD) in which the inverter output voltage harmonics are weighted inversely with the harmonic frequency so as to approximate the current distortion in an inductive load. However, in the large majority of practical applications the load is an induction motor whose impedance does not simply vary linearly with frequency. In this paper a new performance index is proposed which quantifies *harmonic distortion* of the ac current waveform when the inverter supplies a

squirrel cage induction motor load. This new index can be used as an improved means of comparing various modulation algorithms when supplying a squirrel cage induction motor load.

2. The Harmonic Voltage Distortion Factor

Given that the output voltage of a solid state converter $v(t)$ is a periodic function with period T , the RMS value of the function is, by definition,

$$V_{rms} = \sqrt{\frac{1}{T} \int_0^T v(t)^2 dt} \quad (1)$$

Since $v(t)$ is periodic with no dc component, then it can be represented by the Fourier Series,

$$v(t) = V_1 \cos \omega_1 t + V_2 \cos 2\omega_1 t + V_3 \cos 3\omega_1 t + \dots \quad (2)$$

whereupon

$$V_{rms} = \sqrt{\frac{1}{T} \int_0^T \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} V_h V_k \cos(h\omega_1 t) \cos(k\omega_1 t) dt} \quad (3)$$

Upon expanding, the integration of terms in which $h \neq k$ become zero, so that, finally

$$V_{rms} = \sqrt{\sum_{h=1}^{\infty} \frac{V_h^2}{2}} \quad (4)$$

or, in terms of RMS values of the harmonics,

$$V_{rms} = \sqrt{\sum_{h=1}^{\infty} V_{h,rms}^2} \quad (5)$$

In most practical cases the fundamental component can be considered as the desired output. The remainder of this expression is then considered as the “distortion”. Factoring out the desired component

$$V_{rms} = V_{1,rms} \sqrt{1 + \sum_{h=2}^{\infty} \left(\frac{V_{h,rms}}{V_{1,rms}}\right)^2} \quad (6)$$

The *total harmonic distortion (THD)* of the voltage can now be defined as

$$THD = \frac{V_{rms}}{V_{1,rms}} = \sqrt{\sum_{h=2,3,\dots}^{\infty} \left(\frac{V_{h,rms}}{V_{1,rms}}\right)^2} \quad (7)$$

Since the RMS values of both contain a $\sqrt{2}$ factor, the THD is equally expressed in terms of peak values, i.e.

$$THD = \sqrt{\sum_{h=2,3,\dots}^{\infty} \left(\frac{V_h}{V_1}\right)^2} \quad (8)$$

3. Harmonic Current Distortion Factor (WTHD)

While it is the output voltage which is the quantity that is controlled in a voltage source/stiff inverter type system, it is the current which is frequently of most interest since losses, output power etc. typically involve this quantity rather than voltage directly.

Defining the *current harmonic distortion* in the same manner as for voltage, one can state that

$$HD_i = \sqrt{\frac{\sum_{h=2,3,\dots}^{\infty} (I_h)^2}{I_1^2}} \quad (9)$$

The current waveform is, of course, dependent upon the load impedance and, as such, can not be predicted or characterized in advance. However, in some applications, the load can be characterized by a lossy inductance, that is, by an inductance in which the resistance is relatively small. In this case, the har-

monic current amplitudes can be approximated by the expression,

$$I_h \cong \frac{V_h}{h\omega_1 L} \quad h = 2,3,4,\dots \quad (10)$$

where ω is the angular frequency of the fundamental component of the current waveform. If dc components do not exist

$$HD_i = \frac{1}{\omega_1 L} \sqrt{\sum_{h=2}^{\infty} \left(\frac{V_h}{h}\right)^2} \quad (11)$$

Normalizing this expression to the quantity $V_1/(\omega_1 L)$ the weighted total harmonic distortion (*WTHD*) becomes defined as

$$WTHD = \frac{\sqrt{\sum_{h=2}^{\infty} \left(\frac{V_h}{h}\right)^2}}{V_1} \quad (12)$$

4. The Induction Motor Load

The *WTHD* is superior to the *THD* as a figure of merit for a non-sinusoidal converter waveform since the *WTHD* predicts the distortion in the current and subsequent additional losses which are typically the major issues in the application of such converters. However, it should be recalled that Eq. (12) was derived assuming that the load resistance and inductance were constant. When the load is passive, this assumption is generally quite valid. However, when used with a motor load special precautions must be taken because of the non-linear nature of the load impedance. The assumption of constant inductance and resistance is again valid for the stator circuit of any random wound ac machine. However, the shorted bars of the rotor of an induction motor produce special problems due to *deep bar effect* which causes the rotor current to rise to the air gap side of the rotor bars. The overall result is that the resistance and inductance of the rotor circuit become frequency dependent.

5. Rectangular Squirrel Cage Bar [1]

Consider the simple rectangular bar placed in an iron slot as shown in Figure 1. The total *IR* drop across the length of the bar at any height *y* can be obtained by integrating the electric field over the length of the bar.

$$\tilde{V}_R = \int_0^l \tilde{E} \cdot dl \quad (13)$$

where l denotes the length of the rotor bar in the laminations. It is shown in [1] that the voltage drop reduces to

$$\tilde{V}_R = \frac{\gamma \rho I_m l}{b} \left[\frac{\cosh(\gamma y)}{\sinh(\gamma d)} \right] \quad (14)$$

where,

$$\gamma = \sqrt{\frac{j\omega_b \mu_o}{\rho}} \quad (15)$$

ω_b is the angular frequency of the EMF impressed on the rotor bar, μ_o is the permeability of air and ρ is the resistivity of the conducting bar. Note that the resistive drop is a maximum at the top of the slot where the parameter $y = d$.

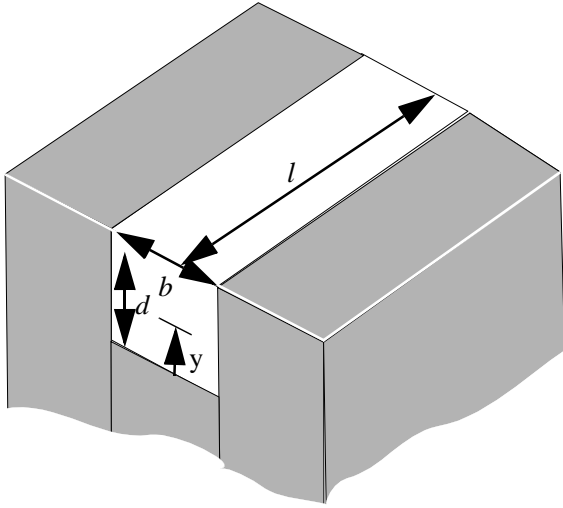


Figure 1 Rotor bar embedded in laminations

The total flux crossing the slot above the height y which therefore links the current below y is

$$\tilde{\phi}_m(y) = \int_0^l \int_0^d \tilde{B}_m dy dz \quad (16)$$

or

$$\tilde{\phi}_m(y) = \frac{\mu_o I_m l}{\gamma b_s} \left[\frac{\cosh(\gamma d) - \cosh(\gamma y)}{\sinh(\gamma d)} \right] \quad (17)$$

The total IX drop across the length of the bar at any height y is therefore

$$\tilde{V}_L = \frac{\partial}{\partial t} \int_0^l \int_0^d \tilde{B}_m dy dz \quad (18)$$

or in complex form

$$\tilde{V}_L = j\omega_b \tilde{\phi}_m \quad (19)$$

whereupon

$$\tilde{V}_L = \frac{j\omega_b \mu_o I_m l}{\gamma b_s} \left[\frac{\cosh(\gamma d) - \cosh(\gamma y)}{\sinh(\gamma d)} \right] \quad (20)$$

Note that the reactive drop is zero at the top of the slot and a maximum at the bottom of the slot, just the reverse of the resistive drop.

The total voltage drop along the bar at an arbitrary height y is the sum of the inductive and resistive drop or

$$\tilde{V}_{bar} = \tilde{V}_R + \tilde{V}_L \quad (21)$$

$$= \frac{\gamma \rho I_m l}{b} \left[\frac{\cosh(\gamma y)}{\sinh(\gamma d)} \right] + \frac{j\omega_b \mu_o I_m l}{\gamma b_s} \left[\frac{\cosh(\gamma d) - \cosh(\gamma y)}{\sinh(\gamma d)} \right] \quad (22)$$

which can also be written as [148],

$$\tilde{V}_{bar} = \frac{\gamma \rho I_m l}{b} \frac{\cosh(\gamma d)}{\sinh(\gamma d)} \quad (23)$$

Note that the total drop down the length of the bar is constant, which is to be expected since all of the current filaments are “connected” in parallel. The current distribution in the bar may be thought of as the superposition of a uniform (average) current and a circulating current flowing additively at the top of the bar and negatively at the bottom, directed in such a way as to oppose the time rate of change of slot leakage flux. The skin effect forces current to flow in the part of the conductor area which lies at the top of the slot and the effective resistance is many times (usually three to four times) as large as the round bar or shallow bar rotor. In the case of the double bar rotor, the resistance of the top bar usually has a higher resistance than the bottom bar making the effective resistance at starting even larger if desired.

Equation (23) can be written in terms of the dc resistance of the bar if it is recalled that for a rectangular bar

$$R_{DC} = \frac{\rho l}{bd} \quad (24)$$

Hence, the effective impedance of the bar is

$$\tilde{Z}_{bar} = \frac{\tilde{V}_{bar}}{I_m} \quad (25)$$

$$= R_{DC} \left[\frac{(\gamma d) \cosh(\gamma d)}{\sinh(\gamma d)} \right] \quad (26)$$

The real portion of the impedance representing the ac resistance of the bar can be readily evaluated as

$$R_{AC} = \alpha d R_{DC} \left[\frac{\sinh(2\alpha d) + \sin(2\alpha d)}{\cosh(2\alpha d) - \cos(2\alpha d)} \right] \quad (27)$$

where

$$\alpha = \sqrt{(\omega_b \mu_o) / (2\rho)} \quad (28)$$

The imaginary component of the impedance represents the reactance of the bar and is

$$X_{AC} = \alpha d R_{DC} \left[\frac{\sinh(2\alpha d) - \sin(2\alpha d)}{\cosh(2\alpha d) - \cos(2\alpha d)} \right] \quad (29)$$

The inductance of a rectangular bar in a slot is [148],

$$L_{DC} = \mu_o \frac{dl}{3b} \quad (30)$$

so that (29) can also be written as,

$$X_{AC} = \frac{3\omega_b L_{DC}}{2\alpha d} \left[\frac{\sinh(2\alpha d) - \sin(2\alpha d)}{\cosh(2\alpha d) - \cos(2\alpha d)} \right] \quad (31)$$

The ac inductance is therefore,

$$L_{AC} = \frac{3L_{DC}}{2\alpha d} \left[\frac{\sinh(2\alpha d) - \sin(2\alpha d)}{\cosh(2\alpha d) - \cos(2\alpha d)} \right] \quad (32)$$

If αd is large Eqs. (27) and (29) can be approximated by

$$R_{AC} = \alpha d R_{DC} \quad (33)$$

and

$$L_{AC} = \frac{3L_{DC}}{2\alpha d} \quad (34)$$

Equation (33) can, by use of Eq. (24), be expressed in the form

$$\begin{aligned} R_{AC} &= \frac{\rho l}{bd} (\alpha d) \\ &= \rho \left(\frac{l_r}{b(1/\alpha)} \right) \end{aligned} \quad (35)$$

Hence, for sufficiently high frequencies, the ac resistance can be calculated in the same manner as for the dc resistance if one replaces the actual depth of the bar d by an equivalent depth $1/\alpha$. The quantity $1/\alpha$ is called the *skin depth* and the fact that the current distributes itself unevenly over a conductor due to sinusoidal excitation is called the *skin effect*. A plot of R_{AC} and L_{AC} normalized with respect to their dc values is given in Figure 2.

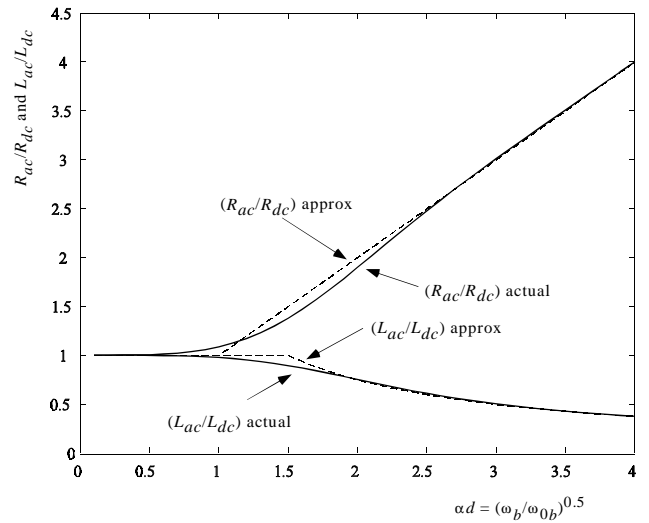


Figure 2 Sketch of normalized resistance and inductance of solid bar in a slot as a function of αd and their approximations

One can write Eqs. (33) and (34) directly in terms of frequency if it is noted that

$$\alpha d = \sqrt{(\omega_b \mu_o) / (2\rho)} d = \sqrt{\omega_b \mu_o d^2 / (2\rho)} \quad (36)$$

Defining a “bar” or “break” frequency

$$\omega_{b0} = \frac{2\rho}{\mu_o d^2} \quad (37)$$

one can also write R_{AC} and L_{AC} for large αd (high frequency) as,

$$R_{AC} = \sqrt{\frac{\omega_b}{\omega_{0b}}} R_{DC} \quad (38)$$

$$L_{AC} = \frac{3}{2} \frac{L_{DC}}{\sqrt{\omega_b/\omega_{0b}}} \quad (39)$$

If the dc value of R_{AC} and L_{AC} are assumed below $\omega_b/\omega_{0b} = 1$ and $9/4$ respectively and high frequency approximations are used above these values the dashed lines of Figure 2 are obtained. A good approximation is obtained which can be used to estimate the harmonic distortion caused by the frequency dependant rotor bar parameters.

5.1 Non-Rectangular Rotor Bars

It is well known that the bar shape of a practical induction motor design is rarely a simple rectangular structure as considered in Section 5. Indeed bar shapes have been devised so as to minimize the losses due to inverter harmonics. Since the analysis of the previous section was limited to this case, it is appropriate to question the validity of the results for non-rectangular bars. While a variety of shapes are possible, it appears that essentially all bar configurations possess a bar break frequency of the type identified as Eq. (37). For example Figure 3 shows the frequency dependence of an inverted coffin shaped bar [2].

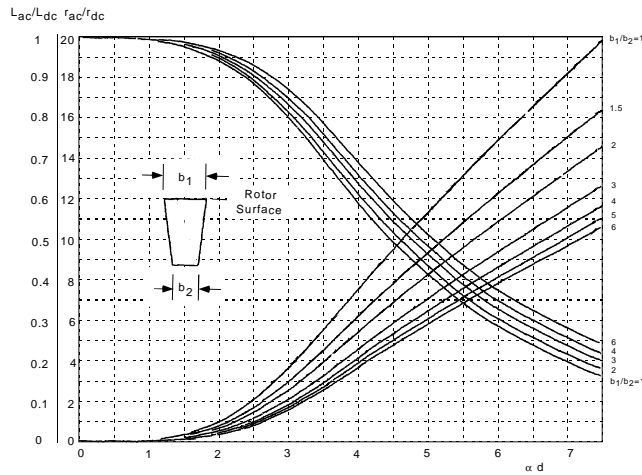


Figure 3 Variation of bar resistance and inductance for inverted coffin shaped bar with αd where d is the bar depth, parameter is b_1/b_2

The frequency behavior of the rotor resistance and leakage inductance can typically be approximated as a constant below this break point and as proportional (i.e. R_{AC}) or inversely proportional (L_{AC}) to $\sqrt{\omega_b/\omega_{0b}}$ beyond this point. The bar shapes can be shown to mainly affect the slope of the curves beyond the break point. However, the analysis that has been

performed can always be modified to accommodate a different slope beyond the break point. For example, let K_b denote the multiplier which changes the slope from a unity value described in Eqs. (38) and (39) obtained for example by finite element analysis. Above the break frequency the AC resistance for the modified case can then be expressed as

$$R_{AC}' = \left[K_b \left(\sqrt{\frac{\omega_b}{\omega_{b0}}} - 1 \right) + 1 \right] R_{DC} \quad (40)$$

$$= K_b R_{AC} + (1 - K_b) R_{DC} \quad (41)$$

Similarly

$$L_{AC}' = \frac{3}{2} \frac{L_{DC}}{K_b \left(\sqrt{\frac{\omega_b}{\omega_{b0}}} - 1 \right) + 1} \quad (42)$$

$$= \frac{\sqrt{\frac{\omega_b}{\omega_{b0}}} L_{AC}}{K_b \left(\sqrt{\frac{\omega_b}{\omega_{b0}}} - 1 \right) + 1} \quad (43)$$

Hence, when the AC resistance and inductance for a particular bar harmonic has been calculated for a simple rectangular bar, the equivalent result can be readily translated to a non-rectangular bar by Eqs. (41) and (43).

6. Per Phase Equivalent Circuit

Although the impedance has been determined for a single bar, all of the bars of an induction motor experience the same impressed flux (only with different phase relationships) so that the motor equivalent circuit rotor impedance maintains the same frequency dependence. Although a portion of the rotor circuit is comprised of the end ring portion, again the same frequency dependence exists. Since the rotor can, for practical purposes, be considered as rotating near synchronous speed, differing only by a small slip, it can be assumed that the bar frequency is related to an arbitrary harmonic impressed on the stator by

$$\omega_b = \omega_h \pm \omega_1 \quad (44)$$

where the plus sign is taken when the stator harmonic rotates in the negative direction, (i.e. 2,5,8,11,...) and the minus sign is used when the harmonic rotates in the positive direction (i.e. 1,4,7,10,13,...). The fre-

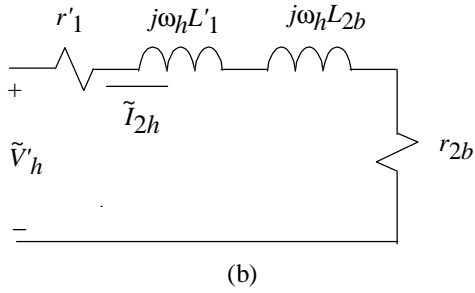
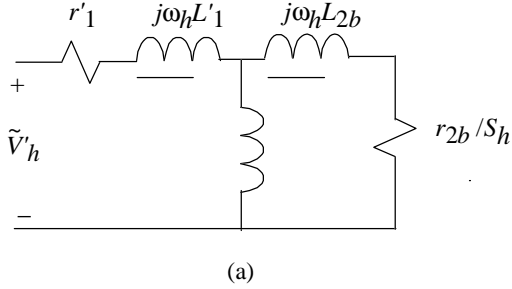
quency dependant rotor resistance and inductance can then be well approximated by,

$$R_{AC} = \sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}} R_{DC} \quad (45)$$

and

$$L_{AC} = \frac{3}{2} \frac{L_{DC}}{\sqrt{(\omega_h \pm \omega_1)/\omega_{0b}}} \quad (46)$$

The resulting per phase equivalent circuit for an induction motor with frequency dependent parameters is shown in Figure 4(a).



$$r_{2h} = r_2 \sqrt{(\omega_h \pm \omega_1)/\omega_{0b}} \quad L_{2h} = (3/2) \frac{L_2}{\sqrt{[(\omega_h \pm \omega_1)/\omega_{0b}]}}$$

$$r_1' = r_1 \frac{L_m}{L_1 + L_m} \quad L_1' = \frac{L_1 L_m}{L_1 + L_m} \quad \tilde{V}_h' = \frac{L_m}{L_1 + L_m} \tilde{V}_h$$

Figure 4 (a) Frequency dependant induction motor equivalent circuit for non-triplen harmonic component and (b) approximation with stator side referred to the rotor

When high frequency harmonics are superimposed on the fundamental, superposition principles can be applied assuming that saturation is not too severe. When the frequency is high the slip frequency corresponding to an arbitrary non-triplen harmonic is

$$S_h = \frac{(\pm\omega_h) - \omega_r}{(\pm\omega_h)} \rightarrow 1 \quad (47)$$

where ω_r is the rotor speed in electrical radians per second and either the plus or minus sign applies depending upon whether the harmonic is a positive or negative sequence respectively. If ω_h is sufficiently high this expression approaches unity regardless of the polarity of ω_h .

Because the rotor parameters are frequency dependant it is useful to refer the stator side of the circuit to the rotor rather than vice versa. Using Thevenin's Theorem, the voltage observed at the air gap from the rotor side is

$$V_h' = \frac{j\omega_h L_m}{r_1 + j\omega_h(L_1 + L_m)} V_h \quad (48)$$

For sufficiently high frequencies, Eq. (48) can be conveniently approximated as

$$V_h' \cong \frac{L_m}{L_1 + L_m} V_h \quad (49)$$

The Thevenin driving point impedance is obtained by shorting the voltage source in which case

$$Z_{1h}' = \frac{(r_1 + j\omega_h L_1)j\omega_h L_m}{r_1 + j\omega_h(L_1 + L_m)} \quad (50)$$

Again, if the angular frequency ω_h is sufficiently high,

$$Z_{1h}' \cong r_1 \frac{L_m}{L_1 + L_m} + j\omega_h \left(\frac{L_1 L_m}{L_1 + L_m} \right) \quad (51)$$

It is clear that the frequency dependence of the rotor parameters must reflect the fact that the rotor is rotating so the fifth harmonic stator voltage depending upon its direction of rotation, impresses a fourth or sixth harmonic voltage on the rotor due to the fact that it is positive or negatively rotating with respect to the rotor. In general, for harmonics related to the fundamental component, the three phase voltage waveforms remain "balanced". That is, the three voltages have identical waveshapes but are only phase shifted in time by 1/3 of a complete cycle. The rotation sequence can be determined by dividing the harmonic number h by 3 and examining the numerator of the resulting fraction (if any). If the numerator is one the sequence is positive and the minus sign in applies in Eq. (47) while if the numerator of the fraction is two the sequence is negative (the plus sign applies). If the value of h is divisible by three the component corre-

sponds to zero sequence term and should be omitted from the summation since this component does not link the rotor.

7. WTHD for Frequency Dependant Rotor Resistance

A very simple figure of merit for the effect of impressing a non-sinusoidal voltage waveform on the rotor branch can be developed if one considers only the frequency dependence of the rotor resistance and assumes that the rotor inductance remains constant. In this case, referring to Eq. (38) and Figure 2, one can approximate the variation of r_2 with frequency as

$$r_{2h} = r_2 \text{ when } \omega_b \leq \omega_{0b}$$

and

$$r_{2h} = r_2 \sqrt{\frac{\omega_b}{\omega_{0b}}} \text{ when } \omega_b > \omega_{0b}.$$

In the first case the loss in the rotor resistance due to a single harmonic is simply

$$P_{2h} = 3I_{2h}^2 r_2 \cong 3 \left(\frac{V_h'}{\omega_h(L_1' + L_2)} \right)^2 r_2 \quad (52)$$

if stator resistance can be neglected. When normalized to the loss occurring during the starting period (due to in-rush current),

$$\frac{P_{2h}}{P_{2, \text{inrush}}} = \frac{3 \left(\frac{V_h'}{\omega_h(L_1' + L_2)} \right)^2 r_2}{3 \left(\frac{V_1'}{\omega_1(L_1' + L_2)} \right)^2 r_2} \quad (53)$$

$$= \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{\omega_1}{\omega_h} \right)^2 \quad (54)$$

which reduces finally to

$$\frac{P_{2h}}{P_{2, \text{inrush}}} = \left(\frac{1}{h} \right)^2 \left(\frac{V_h'}{V_1'} \right)^2 \quad (55)$$

If it can be assumed that the stator resistance drop is negligible compared to the drop across the stator leakage inductance at the frequencies of interest.

When $\omega_b > \omega_{0b}$,

$$P_{2h} = \left(\frac{V_h'}{\omega_h(L_1' + L_2)} \right)^2 r_2 \sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}} \quad (56)$$

$$\frac{P_{2h}}{P_{2, \text{inrush}}} = \frac{\left(\frac{V_h'}{\omega_h(L_1' + L_2)} \right)^2 r_2 \sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}}}{\left(\frac{V_1'}{\omega_1(L_1' + L_2)} \right)^2 r_2} \quad (57)$$

$$\begin{aligned} &= \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{\omega_1}{\omega_h} \right)^2 \sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}} \\ &= \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{\omega_1}{\omega_h} \right)^2 \sqrt{\frac{\omega_h \omega_1}{\omega_1 \omega_{0b}} \pm \frac{\omega_1}{\omega_{0b}}} \\ &\cong \sqrt{\frac{\omega_1}{\omega_{0b}}} \left(\frac{V_h'}{V_1'} \right)^2 \frac{\sqrt{h \pm 1}}{h^2} \quad (58) \end{aligned}$$

One can now define a weighted harmonic distortion factor which includes two sets of terms such that

$$WTHD_{bar} = \sqrt{\sum_{h=2}^{h_0} \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{1}{h} \right)^2 + \sum_{h_0+1}^{\infty} \sqrt{\frac{\omega_1}{\omega_{0b}}} \left(\frac{V_h'}{V_1'} \right)^2 \frac{\sqrt{h \pm 1}}{h^2}} \quad (59)$$

where h_0 denotes the highest harmonic for which $\omega_b < \omega_{0b}$.

It is apparent that the geometry dependent parameter ω_1/ω_{0b} has an important effect on the *WTHD* since the second group of terms are weighted more heavily with respect to frequency than the first group. In most practical cases where pulse width modulation is used, the lower frequency harmonics are suppressed and all of the harmonics are greater than ω_{0b} in which case one can define

$$WTHD_{bar} = \left(\frac{\omega_1}{\omega_{0b}} \right)^{\frac{1}{4}} \sqrt{\sum_{h=2}^{\infty} \left(\frac{V_h'}{V_1'} \right)^2 \frac{\sqrt{h \pm 1}}{h^2}} \quad (60)$$

The quantity ω_1/ω_{0b} can be considered as a ‘‘bar factor’’ related to the design of the machine over which the inverter specialist has little control whereas the remaining portion is most relevant for the purpose of selecting a PWM algorithm. For 60 Hz operation the value of ω_1/ω_{0b} takes on values ranging from 0.25 for small machines (5 HP) to 4.0 for large machines (500 HP).

8. *WTHD* Also Including Effect of Frequency Dependant Rotor Leakage Inductance

Thus far, only the effects of frequency dependant rotor resistance has been considered. When the variation in rotor leakage inductance is also taken into account, the procedure is similar. In this case three regions can be identified,

$$r_{2h} = r_2; L_{2h} = L_2 \text{ when } \omega_b \leq \omega_{0b} \quad (61)$$

$$r_{2h} = r_2 \left(\sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}} \right); L_{2h} = L_2 \quad (62)$$

$$\text{when } \omega_{0b} < \omega_b \leq \frac{9}{4}\omega_{0b}$$

$$r_{2h} = r_2 \left(\sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}} \right); L_{2h} = \frac{(3/2)L_2}{\sqrt{(\omega_h \pm \omega_1)/\omega_{0b}}} \quad (63)$$

$$\text{when } \omega_b > \frac{9}{4}\omega_{0b}$$

The solution for the contributions to the *WTHD* from the first two ranges of ω_h is the same previous work in Section 7.. A typical loss term for a harmonic belonging to the third region is,

$$P_{2h} = \left(\frac{V_h'}{\omega_h \left(L_1' + 1.5 \frac{L_2}{\sqrt{(\omega_h \pm \omega_1)/\omega_{0b}}} \right)} \right)^2 r_2 \sqrt{\frac{\omega_h \pm \omega_1}{\omega_{0b}}} \quad (64)$$

or,

$$P_{2h} = \left(\frac{V_h'}{\omega_h L_1' \left(\sqrt{(\omega_h \pm \omega_1)/\omega_{0b}} + 1.5 \frac{L_2}{L_1'} \right)} \right)^2 r_2 \left(\frac{\omega_h \pm \omega_1}{\omega_{0b}} \right)^{\frac{3}{2}} \quad (65)$$

and when normalized to the power dissipated during the current in-rush interval (assuming constant parameters under this condition),

$$\frac{P_{2h}}{P_{2, \text{inrush}}} = \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{1}{h} \right)^2 \frac{\left(1 + \frac{L_2}{L_1'} \right)^2}{\left[\left(\sqrt{(\omega_h \pm \omega_1)/\omega_{0b}} + 1.5 \frac{L_2}{L_1'} \right) \right]^2} \times$$

$$\left(\frac{\omega_h \pm \omega_1}{\omega_{0b}} \right)^{\frac{3}{2}} \quad (66)$$

Some simplification is possible if it is assumed that the primed stator and rotor leakage inductances are equal, a standard assumption in induction machine analysis. Again neglecting the effects of stator resistance on the voltage ratio V_h'/V_1' , Eq. (66) becomes,

$$\frac{P_{2h}}{P_{2, \text{inrush}}} = \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{1}{h} \right)^2 \frac{4}{\left[\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{(h \pm 1) + 1.5} \right]^2} \left(\frac{\omega_1}{\omega_{0b}} \right)^{\frac{3}{2}} (h \pm 1)^{\frac{3}{2}} \quad (67)$$

A weighted THD factor, termed *WTHD2*, can now be developed as the square root of three sets of terms in which Eq. (54) is used in the first region where $\omega_b \leq \omega_{0b}$, Eq. (58) applies for harmonics in which $\omega_{0b} < \omega_b \leq (9/4)\omega_{0b}$ and Eq. (67) is applicable for values of harmonics for which $\omega_b > (9/4)\omega_{0b}$. Again, the *WTHD* must be presented as function of the machine dependant parameter ω_1/ω_{0b} rather than as a single number. The resulting function is plotted in Figure 5 for the case of the quasi-square wave inverter.

The complicated nature of *IMWTHD2* prompts one to consider acceptable approximations. In most cases the harmonic content of the lower harmonics are negligible, indeed this result is the goal of most pulse width modulation algorithms. In such cases it can be assumed that the harmonic components of flux rotate in the air gap at a sufficiently high rate of rotation that the rotor appears as essentially stationary. Hence, one can, in effect replace the bar frequency $b = h \pm 1$ terms in simply by h . The final result for *IMWTHD2* is

$$IMWTHD2 = \sqrt{W1 + W2 + W3} \quad (68)$$

where

$$W1 = \sum_{\substack{h=3k \pm 1 \\ k=1,2,3,\dots}}^{h_{0b}} \left(\frac{V_h'}{V_{dc}} \right)^2 \left(\frac{1}{h} \right)^2 \quad (69)$$

$$W2 = \sum_{h_{0b}+1}^{h_{2b}} \sqrt{\frac{\omega_1}{\omega_{0b}}} \left(\frac{V_h'}{V_1'} \right)^2 \frac{1}{h^{(3/2)}} \quad (70)$$

$$W3 = \sum_{h_{2b}+1}^{\infty} \left(\frac{V_h}{V_{dc}} \right)^2 \left(\frac{1}{h} \right)^2 \frac{4}{\left[\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{h+1.5} \right]^2} \left(\frac{\omega_1}{\omega_{0b}} \right)^{\frac{3}{2}} (h)^{\frac{3}{2}} \quad (71)$$

In these equations h_{0b} represents the highest harmonic for which $\omega_b < \omega_{0b}$ and h_{2b} denotes the highest harmonic for which $\omega_b < (9/4)\omega_{0b}$. Included in this result, also is the fact that third harmonic components (zero sequence components) of stator current do not link the rotor if the stator winding is an ideal sinusoidally distributed winding. Hence, only values of $h = 2, 4, 5, 7, 8, 10$, etc. are included in the summation. Since the parameter ω_1/ω_{0b} can not be isolated from the remainder of the expression, the WTHD2 is best portrayed as a function of ω_1/ω_{0b} or specified for particular values of ω_1/ω_{0b} .

The exact and approximate functions are plotted in Figure 5 for the case of the quasi-square wave inverter. Since it can be recalled that the quasi-square wave inverter contains a full measure of all non-triplen odd harmonics, the approximation can be considered as sufficiently accurate for use as a figure of merit.

Note that at most only two harmonics can exist in the middle region and could easily be approximated by relegating the lower of the two harmonics to the first region and the upper one to region three. Also, if all of the harmonics are sufficiently higher than ω_{0b} then only values in the third region apply and furthermore if

$$\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{(h \pm 1)} \gg 1.5$$

Eq. (67) simplifies to

$$\frac{P_{2h}}{P_{2, inrush}} = 4 \sqrt{\frac{\omega_1}{\omega_{0b}}} \left(\frac{V_h}{V_1} \right)^2 \frac{\sqrt{h \pm 1}}{h^2} \quad (72)$$

which is simply four times that of Eq. (58). Again making the assumption that the bar frequencies are sufficiently high that $b \cong h$, the WTHD for this condition is

$$IMWTHD2(hf) = 2 \left(\frac{\omega_1}{\omega_{0b}} \right)^{1/4} \sqrt{\sum_{\substack{h=3k \pm 1 \\ k=1,2,\dots}}^{\infty} \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^{3/2}} \quad (73)$$

This result is to be expected since Eq. (72) effectively assumes that the rotor leakage inductance L_{2h} is zero and, since it has been assumed that $L_1' = L_2$, the current for all harmonics will be twice the value obtained when L_{2h} is assumed as a constant equal to L_1' .

It must be mentioned here that the assumption of ideal sinusoidally distributed windings is a convenient mathematical artifice which is never realized exactly in practice. In reality, the stator winding distribution is non-ideal meaning the higher harmonic fields are set up in the air gap, that is, fields with sets of poles which are an odd multiple of the number of poles set up by the fundamental component. In most practical cases a small third harmonic spatial component will therefore exist in the air gap when third harmonic stator currents are allowed to flow, so that a field is produced which does link the rotor bars. This field can be shown to be single phase in nature, i.e. stationary and pulsating in space. Since this effect is generally small, it is neglected here.

9. WTHD for Stator Copper Losses

The equivalent circuit of Figure 4 can also be used to obtain a suitable figure of merit for stator losses. The ratio of stator resistive losses due to harmonic h as a per unit of power dissipated during the in-rush period with fundamental frequency voltage applied is,

$$\frac{P_{1h}}{P_{1, inrush}} = \frac{\left(\frac{V_h'}{\omega_h(L_1' + L_2)} \right)^2 r_1}{\left(\frac{V_1'}{\omega_1(L_1' + L_2)} \right)^2 r_1} \quad (74)$$

When $\omega_1 < \omega_{0b}$ and also when $\omega_{0b} \leq \omega_1 \leq (9/4)\omega_{0b}$ both the stator resistance and the rotor leakage inductance are constant and this expression reduces to

$$\frac{P_{1h}}{P_{1, inrush}} = \left(\frac{V_h'}{V_1'} \right)^2 \left(\frac{\omega_1}{\omega_h} \right)^2 \quad (75)$$

Hence, for harmonics, $0 \leq \omega_b \leq (9/4)\omega_{b0}$, the WTHD for the stator is the same as previously obtained for the rotor when r_2 and L_2 are constant, namely

$$IMWTHD1 = \sqrt{\sum_{h=2}^{\infty} \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^2} \quad (76)$$

For operation above $(9/4)\omega_{0b}$,

$$\frac{P_{1h}}{P_{1, \text{inrush}}} = \frac{\left(\frac{V_h'}{\omega_h \left(L_1' + 1.5 \frac{L_2}{\sqrt{(\omega_h \pm \omega_1)/\omega_{0b}}} \right)} \right)^2 r_1}{\left(\frac{V_1'}{\omega_1 (L_1' + L_2)} \right)^2 r_1} \quad (77)$$

which, after some manipulation and the usual assumptions becomes

$$\frac{P_{1h}}{P_{1, \text{inrush}}} = \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^2 \frac{4}{\left[\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{(h \pm 1) + 1.5} \right]^2} \left(\frac{\omega_1}{\omega_{0b}} \right)^{(h \pm 1)} \quad (78)$$

Again if it is assumed that the bar frequency for each harmonic is sufficiently close to the corresponding stator harmonic frequency, then

$$\frac{P_{1h}}{P_{1, \text{inrush}}} \cong \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^2 \frac{4}{\left[\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{(h) + 1.5} \right]^2} \left(\frac{\omega_1}{\omega_{0b}} \right) \quad (79)$$

A suitable expression for the stator *WTHD* is consequently

$$IMWTHD1 = \sqrt{\frac{\sum_{h=3k \pm 1}^{h_0} \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^2}{\sum_{h=3k \pm 1}^{h_0} \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^2}} \quad (80)$$

where h_0 is the highest harmonic for which $\omega_b < (9/4)\omega_{b0}$.

Finally, for sufficiently high harmonic content (above $(9/4)\omega_{0b}$)

$$\sqrt{(h \pm 1) \left(\frac{\omega_1}{\omega_{0b}} \right)} \gg 1.5 \quad (81)$$

so that, as a final figure of merit one can reasonably approximate the stator *WTHD* by means of the simple expression

$$IMWTHD1(hf) = 2 \sqrt{\sum_{h=3k \pm 1}^{\infty} \left(\frac{V_h}{V_1} \right)^2 \left(\frac{1}{h} \right)^2} \quad (82)$$

which is simply twice the *WTHD* obtained by ignoring the frequency dependence of the motor parameters. The factor of two is obtained by neglecting, in effect, the rotor leakage inductance. If the inverter waveform has only high frequency harmonic content, this result demonstrates that *WTHD*, as defined by (12) remains a reasonable figure of merit for the stator losses of an induction motor but not for the rotor losses.

Figure 5 shows how the two *WTHD* terms vary as a function of ω_1/ω_{0b} for the case of a three phase quasi-square wave inverter. Also shown for reference is the *WTHD* which does not include the effect of frequency dependant parameters (Eq. (12)). It is

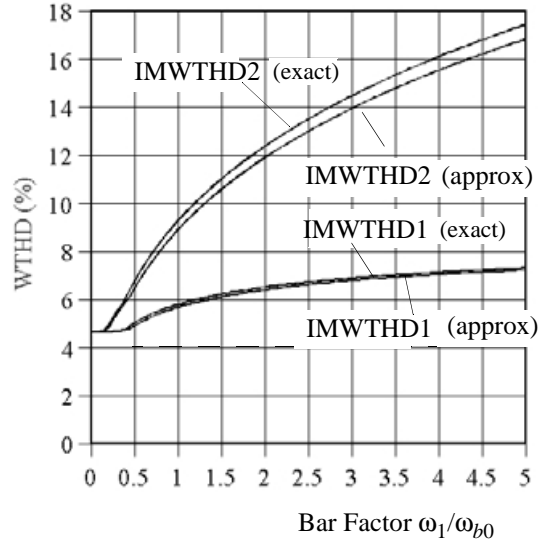


Figure 5 Effect of frequency dependant rotor parameters on the stator and rotor weighted total harmonic distortion

apparent that the deep bar effect has an important effect on the losses, particularly when the fundamental component is substantially greater than the characteristic bar frequency. The effect on the loss is particularly dominant in the rotor since the resistance corresponding to the rotor bar harmonics increase with the bar factor while the stator resistance remains independent of frequency.

10. Example Calculation of Harmonic Losses

The following induction motor is assumed to be powered by a quasi-square wave voltage source inverter;

$$\begin{aligned} \text{HP} &= 50 \text{ HP} & V_{ll,rms} &= 230 \text{ V} \\ \omega_1 &= 377 \text{ r/s} \\ r_1 &= 0.03 \Omega & r_2 &= 0.04 \Omega \\ X_1 &= 0.1234 \Omega & X_2 &= 0.1176 \Omega \\ X_m &= 2.5 \Omega \end{aligned}$$

For the condition of interest, the inverter operates in the field weakening mode at two per unit frequency (based on the rating of the motor). The bar characteristic frequency is determined to be $\omega_{b0} = 300$ r/s. The task is to determine (roughly) the motor losses due to the harmonic content of the inverter.

Solution: Since the angular frequency corresponding to two per unit is

$$\omega_1 = 2(377) = 754$$

the characteristic bar factor is

$$\frac{\omega_1}{\omega_{b0}} = \frac{754}{300} = 2.51$$

The corresponding *WTHDs* from Figure 5 are

$$\begin{aligned} \text{WTHD1} &= 0.067 \\ \text{WTHD2} &= 0.135 \end{aligned}$$

By definition,

$$\frac{\sum_h P_{1h}}{P_{1, \text{inrush}}} = (\text{WTHD1})^2 = 0.067^2 = 0.00449$$

Thus

$$\sum_h P_{1h} = 0.00449 P_{\text{inrush}} = \frac{0.00449(V_1')^2}{[\omega_1(L_1' + L_2)]^2} r_1$$

Now,

$$V_1' = \frac{X_m}{X_m + X_1} V_1 = \frac{2.5}{2.6234} \frac{230}{\sqrt{3}} = 126.5$$

$$\omega_1 L_1' = 754 \frac{L_m L_1}{L_m + L_1} = \frac{2(0.1234)2.5}{0.1234 + 2.5} = 0.2352$$

Thus,

$$\sum_h P_{1h} = \frac{0.00449(V_1')^2}{[\omega_1(L_1' + L_2)]^2} r_1 = \frac{0.00449(126.5^2)}{0.4704^2} 0.03 = 9.74$$

Watts/Phase. Similarly,

$$\frac{\sum_h P_{2h}}{P_{1, \text{inrush}}} = (\text{WTHD2})^2 = 0.135^2 = 0.0182$$

Thus

$$\sum_h P_{2h} = 0.0182 P_{\text{inrush}} = \frac{0.0182(V_1')^2}{[\omega_1(L_1' + L_2)]^2} r_2$$

and,

$$\sum_h P_{2h} = \frac{0.0182(126.5^2)}{0.4704^2} 0.04 = 52.64 \text{ Watts/Phase}$$

The total copper loss due to harmonic content in the quasi-square wave inverter is

$$\sum_h P_h = 3 \left(\sum_h P_{h1} + \sum_h P_{h2} \right) = 187 \text{ Watts}$$

which represents roughly 0.5% of the motor 50 HP rating.

11. THD Weighting with PWM Inverter Supply

In cases where the fundamental component remains constant or when the dc link voltage is proportional to the fundamental ac component as is the case for constant volts/hertz drives, the normalization of the *WTHD* by the fundamental ac component is a suitable choice. In particular, note the *THD* and *WTHD* of a quasi square wave inverter types does not change with frequency provided that the dc voltage decreases in direct proportion. In this case the fundamental component as well as all of the harmonics change by the same value making the *THD* and the *WTHDs* independent of frequency. In the case of a pulse-width modulated drive however, the dc voltage remains constant while the fundamental component varies. On the other hand, the harmonic components change relatively little, assuming the same ratio of switching to output frequency, making the *THD* and

WTHDs vary widely. In fact, it is easy to see that when the fundamental component approaches zero (by virtue of the modulation index M approaching zero), the distortion factors approach infinity, clearly an unsatisfactory situation. The problem can be avoided by simply choosing an normalization factor which is invariant as frequency changes. For the case of the half bridge inverter this quantity is conveniently chosen as the value of ac fundamental ac voltage existing when the modulation index $M = 1$, defined as V_{dc} where V_{dc} equals one-half the pole to pole dc bus voltage. Using the approximation $b = h$, the *WTHDs* normalized to the inverter dc link voltage become

$$IMWTHD0 = \frac{\sqrt{\sum_{h=2}^{\infty} (V_h)^2 \frac{1}{h^2}}}{V_1|_{M=1}} = \sqrt{\sum_{h=2}^{\infty} \left(\frac{V_h}{V_{dc}}\right)^2 \frac{1}{h^2}} \quad (83)$$

$$IMWTHD01 = \sqrt{\sum_{\substack{h=3k \pm 1 \\ k=1,2,3,\dots,k_{0b}}}^{h_{0b}} \left(\frac{V_h}{V_{dc}}\right)^2 \left(\frac{1}{h}\right)^2 + 4 \left(\frac{\omega_1}{\omega_{0b}}\right)^2 \sum_{\substack{h=3k \pm 1 \\ k=k_{0b}+1,\dots}}^{\infty} \left(\frac{V_h}{V_{dc}}\right)^2 \left(\frac{1}{h}\right)^2 \frac{1}{\left[\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{h} + 1.5\right]^2}} \quad (84)$$

$$WTHD02 = \sqrt{W1 + W2 + W3} \quad (85)$$

where

$$W1 = \sum_{\substack{h=3k \pm 1 \\ k=1,2,3,\dots}}^{h_{0b}} \left(\frac{V_h}{V_{dc}}\right)^2 \left(\frac{1}{h}\right)^2 \quad (86)$$

$$W2 = \sum_{h_{0b}+1}^{h_{2b}} \sqrt{\frac{\omega_1}{\omega_{0b}}} \left(\frac{V_h}{V_{dc}}\right)^2 \left(\frac{1}{h}\right)^{\frac{3}{2}} \quad (87)$$

$$W3 = \sum_{h_{2b}+1}^{\infty} \left(\frac{V_h}{V_{dc}}\right)^2 \left(\frac{1}{h}\right)^{1/2} \frac{4}{\left[\sqrt{\frac{\omega_1}{\omega_{0b}}} \sqrt{h} + 1.5\right]^2} \left(\frac{\omega_1}{\omega_{0b}}\right)^{\frac{3}{2}} \quad (88)$$

In these equations h_{0b} again represents the highest harmonic for which $\omega_b < \omega_{0b}$ while h_{2b} denotes the highest harmonic for which $\omega_b < (9/4)\omega_{0b}$. If

desired the *WTHD* is readily obtained from these expressions by taking,

$$IMWTHD = IMWTHD0 \frac{V_{dc}}{V_1} = \frac{IMWTHD0}{M} \quad (89)$$

$$IMWTHD1 = \frac{IMWTHD01}{M} \quad (90)$$

$$IMWTHD2 = \frac{IMWTHD02}{M} \quad (91)$$

12. Conclusion

This paper has developed a new figure of merit for the comparison of the performance inverter pulse width modulation algorithms when feeding a squirrel cage induction motor load. Such a quantifiable figure of merit should prove useful in assessing how various PWM algorithms influence motor copper losses.

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