A Generalized Transformation Methodology for Polyphase Electric Machines and Networks

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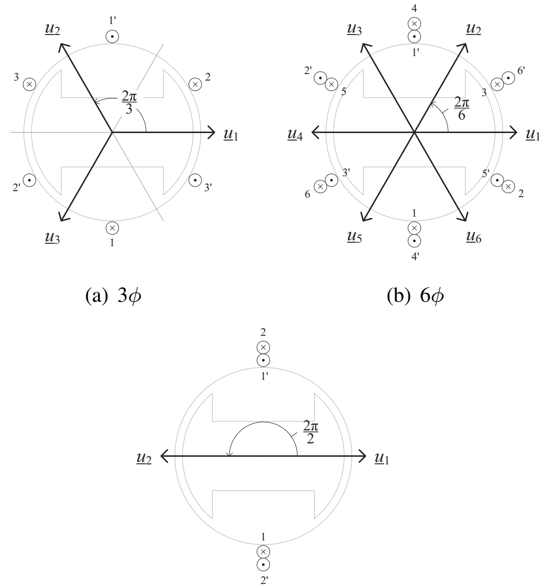
Abstract*—This paper introduces a methodology by which the dq electromagnetic model of an AC machine or network can be extended to a system of any number of phases. The methodology proposed here is based on pole-symmetry (symmetry with respect to π rather than 2π) and separates the electrical configuration from the magnetic configuration of the machine, leading to the concept of the fundamental winding configuration. It is shown that any possible winding configuration can be accounted for using the generic fundamental winding configuration together with a winding configuration matrix. Symmetry with respect to π rather than 2π suggests a change to the complex operator that has served as the basis of Fortescue's method of symmetrical components for nearly 100 years. Using the new complex operator, the authors derive a modified symmetrical component transformation for the fundamental winding configuration. It is shown that when the configuration matrix is applied, the modified transformation yields the same results as the original, but lends itself to a more systematic generalization methodology. Then, following the historical progression, the generalized Clarke and Park transformations are derived from the modified symmetrical component transformation. These transformations presented here enable the systematic generalization of the dq electromagnetic machine model, including the effects of saliency. The generalized dq model, along with laboratory results of a nine-phase permanent magnet synchronous machine, will be presented in a series of follow-up papers.

I. Introduction

Higher-phase order transformations, machine models and results have been reported in the literature before [1]–[15], but they are usually presented for a particular phase-order and/or winding configuration with the perfunctory statement that such transformations and models can be generalized to any number of phases. However there are a number of configuration issues that need to be addressed before one can approach the generalization of modeling, control and modulation techniques. For example, one of the most complete works with regard to generalization was an $n - m$ phase induction machine model put forth by White and Woodson in their 1959 text on electromechanical energy conversion [16]. In their text, they developed a generalized orthogonal transformation and subsequent machine model, but acknowledged that the model did not work for an even number of phases. In that case, they proposed to develop the transformation and model for a machine of $2n$ phases and constrain the extra terminal values to be the negative of the first n phases. The methodology proposed here seeks to circumvent this trick. Furthermore there is another issue with the generalization. As the number of phases increases, so too does the possible phase permutations. For example, a fifteen phase machine could be connected as five three-phase sets, three five-phase sets or a 15-phase set with a single neutral. This problem may arise whenever the phase order is not prime.

In this paper, the authors present a methodology by which the model of an AC salient machine or network can be generalized to any number of phases. The method addresses both the issues of even-ordered machines and winding permutations. It involves separating, from the modeling process, the manner in which the polyphase winding sets are configured. The model is developed based on a fundamental winding configuration which exhibits pole symmetry rather than symmetry about the stator periphery (symmetric with respect to π rather than 2π). The use of this fundamental winding configuration makes it easy to generalize the AC salient machine model to any number of phases–odd or even–and can accommodate any possible winding configuration with the application of a simple configuration matrix. But since it is based on pole symmetry, it calls for a new set of symmetric, orthogonal and rotational transformations. This concept of the fundamental winding configuration and the derivation of the necessary transformations are reported here as a prerequisite to the derivation of the generalized electromagnetic model of an n-phase salient synchronous machine to be reported in a forthcoming paper. The authors believe the proposed methodology and the resulting transformations provide valuable insight to the nature of high-phase-order (HPO) machines. Then, to validate the approach, the balance of the work including generalized control and modulation techniques along with laboratory results of a specially constructed nine-phase salient PM machine will appear in a following paper.

This paper will be organized as follows: In section II of the paper, the authors will introduce the concept of the pole-symmetric fundamental winding configuration. It is shown that any possible winding configuration can be accommodated by the application of a configuration matrix to the base-line (or fundamental) winding configuration. The pole-symmetric configuration suggests that the symmetrical component transformation be based on a different complex operator than the one originally used in Fortescue's method. It is important to note that the Method of Symmetrical Components, with all its



(a) 3ϕ (b) 6ϕ

(c) 2ϕ

Fig. 1. Two-pole symmetric AC machine vector diagrams

contributions to present day analytical methods, is unaltered. The new symmetrical component transformation, based on the pole-symmetric complex operator, is simply better suited for generalization to any number of phases. It is derived in Section III. The reader may find this new transformation to be very insightful with regard to the nature of the extra components (not just positive, negative and zero sequence components) that are present in HPO machines and networks. Then, the ubiquitous Clarke and Park transformations, originally derived from Fortescue's symmetrical component transformation, are revisited with respect to the pole-symmetric version of the symmetrical component transformation in Section IV. These new pole-symmetric transformations are truly generalized with respect to phase order. In section V, it will be shown that the dq model of HPO machines and networks is independent of any particular winding configuration (only the excitation differs). Hence, this methodology based on the pole-symmetric fundamental winding configuration, enables the derivation of a truly generalized dq model of an n-phase salient pole AC synchronous machine.

II. The Fundamental Winding Configuration

A. Two-pole Symmetry

Consider the simplified depiction of a three-phase machine stator shown in Fig. 1(a). It shows several sets of coils distributed symmetrically around the periphery of the two-pole stator (e.g. *two-pole symmetry*). Each coil is comprised of a conductor k and its return conductor k', representing the winding of the k^{th} phase. When a current flows into conductor k and returns out of conductor k', a proportional flux is produced in the direction depicted by the unit vector $\underline{u}_k$. Of course, a coil current flowing in the opposite direction will produce a flux in the direction of $-\underline{u}_k$.

The vector sum of the simultaneous phase flux vectors interacts with the rotor field field flux, often with the goal of producing a constant magnitude rotating stator flux vector at a particular rotational frequency. This is accomplished by exciting the set of symmetrically dispersed phase coils with a set of balanced, periodic phase currents symmetrically phase shifted in time. Hence, the notion of two-pole symmetry is deeply embedded in ac machine and network theory. The complex operator

$$\underline{a} = e^{j\frac{2\pi}{3}} \tag{1}$$

is often used to represent the phase shift between the adjacent magnetic axes of the ac machine as well as the phasors of the voltage, current or flux vectors. The vectors of Fig. 1(a) are depicted as

$$\underline{u}_k = \underline{a}^{(k-1)}\Big|_{k\in\{1,2,3\}}. \tag{2}$$

It seems quite natural to extend this notion of two-pole symmetry in our attempt to generalize AC machine and network theory to any number of phases. In other words, the phase shift of the adjacent components of an n-phase machine or network is represented by the generalized complex operator

$$\underline{a} = e^{j\frac{2\pi}{n}}. \tag{3}$$

Figs. 1(b) and (c) show the vector diagrams of the 6ϕ and 2ϕ machine based on two-pole symmetry. However, both of these machines present somewhat of a problem. In the case of the 6ϕ machine, only three of the six magnetic axes are unique. Coil sets 1/1' and 4/4' share the same magnetic axis. Flux from the one coil either adds or subtracts from that of the other. And indeed, such machines have been shown to exhibit identical characteristics to their three-phase counterparts [2], [6]. Furthermore, the magnetic axes of the two-phase machine in Fig. 1(c) are colinear. The machine is incapable of producing a rotating flux vector. The two-phase machine, based on two-pole symmetry cannot be represented by the ubiquitous two-phase dq equivalent model. This may not be so much of an issue in the case of a particular machine. Indeed, White and Woodson [16], Klingshirn [5], [6] and others have proposed methods to deal with even-phase order with various winding configurations. But it does pose a problem in the development of a generalized electromagnetic machine model that is agnostic to phase-order and winding configuration.

B. Pole Symmetry

As previously stated, when the phase order is even, White and Woodson suggested to double the phase number and constrain the additional phases to be the negative of the first. In this case, the complex operator effectively becomes

$$\underline{a} = e^{j\frac{2\pi}{2n}} = e^{j\frac{\pi}{n}}. \tag{4}$$

This is akin to defining the symmetry of the machine or network according to π electrical radians rather then 2π. This can be referred to as single-pole symmetry, or more simply, *pole-symmetry*. It turns out that this pole-symmetric operator works equally well for any order system–odd or even. One simply needs to define whether the "sense" of the phase axis is positive or negative.

Let the complex operator $\underline{\alpha}$ represent the phase shift between the magnetic axes of a machine according to pole-symmetry

$$\underline{\alpha} = e^{j\frac{\pi}{n}}. \tag{5}$$

Fig. 2 shows the pole-symmetric vector diagrams for the three, six and two phase machines. The magnetic axes are represented by the unit vector

$$\underline{u}_k = p_k \cdot \underline{\alpha}^{(k-1)}\Big|_{k\in\{1,2,3\}}, \tag{6}$$

where p_k is either 1 or -1 depending on the desired "sense" of the winding. One can clearly see that the vector diagram of Fig. 1(a) and that of Fig. 2(a) are identical, aside from some minor differences in notation. In fact, the unit vectors of Fig. 1(a) can be derived from those of Fig. 2(a) by the following relation

$$\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\\ \underline{u}_3\end{bmatrix}_{(2\pi)} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\\ \underline{u}_3\end{bmatrix}_{(\pi)}. \tag{7}$$

Figs. 2(b) and (c) show the pole-symmetric depiction of the six and two-phase machines, respectively. It is clear that the pole-symmetric six-phase machine has six independent magnetic axes. And the pole-symmetric two-phase machine *is* capable of producing a rotating flux vector.

C. Configuration Matrix

The practical winding configuration or "sense" of the winding is accommodated by the use of a configuration matrix $[\mathbf{P}]$, such as in (7) where

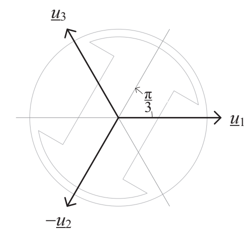
$$[\mathbf{P}] = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix} \tag{8}$$

and

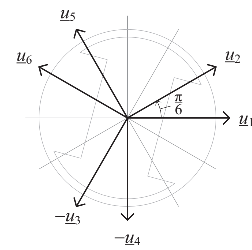
$$\boldsymbol{u}_{k_{(2\pi)}} = [\mathbf{P}]\,\boldsymbol{u}_{k_{(\pi)}}. \tag{9}$$

This leads to the concept of the *fundamental winding configuration*.

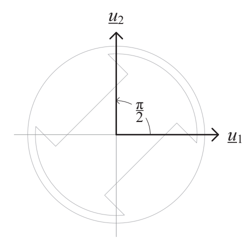
In Fig. 3(a), the common vector diagram (winding configuration) of the 6ϕ ac machine is given. It depicts a machine with two sets of three-phase windings phase shifted by 30 electrical degrees (or $\frac{\pi}{6}$ radians). This is the same vector diagram as that of Fig. 2(b) but with perhaps the more familiar notation. Fig. 3(b) depicts the *fundamental vector diagram* of the 6ϕ machine. It is not difficult to see that the conversion from the fundamental configuration to the practical configuration is



(a) 3ϕ

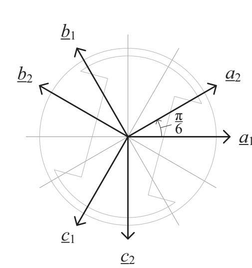


(b) 6ϕ

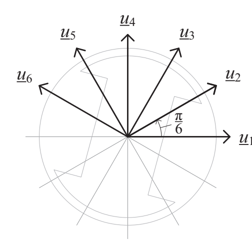


(c) 2ϕ

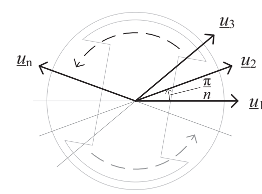
Fig. 2. Pole symmetric vector diagrams



(a) Practical 6ϕ configuration



(b) Fundamental 6ϕ configuration



(c) Fundamental $n\phi$ configuration

Fig. 3. The fundamental winding configuration

accomplished by the following matrix

$$\begin{bmatrix}\underline{a}_1\\ \underline{b}_1\\ \underline{c}_1\\ \underline{a}_2\\ \underline{b}_2\\ \underline{c}_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\\ \underline{u}_3\\ \underline{u}_4\\ \underline{u}_5\\ \underline{u}_6\end{bmatrix} \tag{10}$$

or

$$\boldsymbol{x}_{abc} = [\mathbf{P}]\,\boldsymbol{x}_{123}. \tag{11}$$

In general, let the vector

$$\boldsymbol{x}_{abc} = \left[\underline{a}_1,\, \underline{b}_1,\, \ldots\right]^T \tag{12}$$

represent the the vector elements in the practical (or terminal) coordinates and let

$$\boldsymbol{x}_{123} = \left[\underline{u}_1,\, \underline{u}_2,\, \ldots,\, \underline{u}_n\right]^T \tag{13}$$

represent the vector elements in the fundamental coordinates. Then (11) represents the transformation from fundamental to terminal coordinates and the inverse transformation is given by

$$\boldsymbol{x}_{123} = [\mathbf{P}]^{-1}\,\boldsymbol{x}_{abc}. \tag{14}$$

It is not too difficult to verify that $[\mathbf{P}]^{-1} = [\mathbf{P}]^T$. This can be shown to be true for any configuration matrix with one-to-one mapping.

Finally, Fig. 3(c) represents the $n\phi$ fundamental winding configuration vector diagram. This is the generic $n\phi$ configuration that can take on any of the practical winding configurations deemed suitable for a machine of n phases. For example, the 9 phase fundamental configuration could be transformed into a machine with three 3ϕ sets offset by 20 electrical degrees (usually referred to as a nine-phase asymmetric machine) or with the 3ϕ sets offset by 40 electrical degrees (symmetric machine), it only requires a change in the configuration matrix.

The utility of this methodology will become apparent when, in Section V, it is shown that the dq model itself, does not depend on the winding configuration. Hence, the dq model of the n-phase machine can be developed based on the fundamental winding configuration. Then, the generalized fundamental dq model can be tailored to a particular machine by defining the phase order n and the winding configuration $[\mathbf{P}]$. In order to develop the fundamental model, one first needs to derive the proper pole-symmetric transformations.

III. Symmetrical Component Transformation

A. Original Symmetrical Components

Most engineers skilled in the analysis of ac machines and networks are familiar with Fortescue's Method of Symmetrical Components [17], at least for the three-phase case given below

$$\begin{bmatrix}\underline{y}_0\\ \underline{y}_+\\ \underline{y}_-\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \underline{a} & \underline{a}^2\\ 1 & \underline{a}^2 & \underline{a}\end{bmatrix}\begin{bmatrix}\underline{x}_a\\ \underline{x}_b\\ \underline{x}_c\end{bmatrix}. \tag{15}$$

Using the circular identity of the complex operator

$$\underline{a}^{k\pm xn} = \underline{a}^k, \tag{16}$$

where x and k are integers and $n = 3$ for the 3ϕ case, (15) can be written as

$$\begin{bmatrix}\underline{y}_0\\ \underline{y}_+\\ \underline{y}_-\end{bmatrix} = \frac{1}{3}\begin{bmatrix}\underline{a}^0 & \underline{a}^0 & \underline{a}^0\\ \underline{a}^0 & \underline{a}^1 & \underline{a}^2\\ \underline{a}^0 & \underline{a}^2 & \underline{a}^4\end{bmatrix}\begin{bmatrix}\underline{x}_a\\ \underline{x}_b\\ \underline{x}_c\end{bmatrix}. \tag{17}$$

By letting $\boldsymbol{y}_{123} = \boldsymbol{y}_{0+-}$ and $\boldsymbol{x}_{123} = \boldsymbol{x}_{abc}$, (17) can be written in a more compact and generalized form as

$$\underline{y}_i = \frac{1}{n}\sum_{k=1}^{n} \underline{a}^{(i-1)(k-1)}\underline{x}_k\Big|_{i\in\{1,2,\ldots,n\}}. \tag{18}$$

It can be shown that the inverse transformation is then given by

$$\underline{x}_k = \sum_{i=1}^{n} \underline{a}^{-(i-1)(k-1)}\underline{y}_i\Big|_{k\in\{1,2,\ldots,n\}}. \tag{19}$$

Equation (18) represents the transformation that is the basis of Fortescue's Method of Symmetrical Components. It is generalized for any number of phases. However, as discussed in Section II, when n is even, one half of the phase vectors oppose one another and machines so constructed have been shown to exhibit the same characteristics (harmonics, torque ripple, etc.) as those with half the number of phases. Furthermore, when n is not prime, there may be multiple possible winding configurations. It was shown that both of these issues can be handled by the use of the fundamental winding configuration and a winding configuration matrix.

B. Modified Symmetrical Components

It is not too difficult to prove that, in contrast to that of the two-pole symmetric complex operator $\underline{a}$ given in (16), the circular identity of the pole-symmetric complex operator $\underline{\alpha}$ as defined in (5) is given by

$$\underline{\alpha}^{k\pm xn} = (-1)^x\,\underline{\alpha}^k. \tag{20}$$

As was mentioned in Section II, White and Woodson had proposed to double the number of phases and constrain the extra terminal values to be the negative of the original values

$$\underline{x}_{(n+k)} = -\underline{x}_k\big|_{k\in\{1,2,\ldots,n\}}. \tag{21}$$

Doubling the number of phases led to the pole-symmetric complex operator α as shown in (4) and (5). Applying this methodology to the symmetrical component transformation, (18) can be written as

$$\underline{y}_i = \frac{1}{2n}\sum_{k=1}^{2n} \underline{\alpha}^{(i-1)(k-1)}\underline{x}_k\Big|_{i\in\{1,2,\ldots,2n\}}. \tag{22}$$

Then, by breaking out the additional terminal values separately, the summation limit can be reduced to n and (22) can be written in the following form

$$\underline{y}_i = \frac{1}{2n}\sum_{k=1}^{n}\Big(\underline{\alpha}^{(i-1)(k-1)}\underline{x}_k + \underline{\alpha}^{(i-1)(n+k-1)}\underline{x}_{(n+k)}\Big)\Big|_{i\in\{1,2,\ldots,2n\}}. \tag{23}$$

Now applying the constraint in (21) and doing some simplification, (23) becomes

$$\underline{y}_i = \frac{1}{2n}\sum_{k=1}^{n}\left(1+\underline{\alpha}^{ni}\right)\underline{\alpha}^{(i-1)(k-1)}\underline{x}_k\Big|_{i\in\{1,2,\ldots,2n\}}. \tag{24}$$

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \vdots \\ \underline{x}_{(n-1)} \\ \underline{x}_n \end{bmatrix} = \frac{1}{\sqrt{n}} \left\{ \underbrace{\begin{bmatrix} 1 \\ \underline{\alpha}^{-1} \\ \underline{\alpha}^{-2} \\ \vdots \\ \underline{\alpha}^{-(n-2)} \\ \underline{\alpha}^{-(n-1)} \end{bmatrix}}_{\text{CW fund. sequence}} \underline{y}_1 + \underbrace{\begin{bmatrix} 1 \\ \underline{\alpha}^{-3} \\ \underline{\alpha}^{-6} \\ \vdots \\ \underline{\alpha}^{-(n-6)} \\ \underline{\alpha}^{-(n-3)} \end{bmatrix}}_{\text{CW 3}^{\text{rd}}\text{ harm. sequence}} \underline{y}_2 + \underbrace{\begin{bmatrix} 1 \\ \underline{\alpha}^{-5} \\ \underline{\alpha}^{-10} \\ \vdots \\ \underline{\alpha}^{-(n-10)} \\ \underline{\alpha}^{-(n-5)} \end{bmatrix}}_{\text{CW 5}^{\text{th}}\text{ harm. sequence}} \underline{y}_3 + \cdots + \underbrace{\begin{bmatrix} 1 \\ \underline{\alpha}^{3} \\ \underline{\alpha}^{6} \\ \vdots \\ \underline{\alpha}^{(n-6)} \\ \underline{\alpha}^{(n-3)} \end{bmatrix}}_{\text{CCW 3}^{\text{rd}}\text{ harm. sequence}} \underline{y}_{(n-1)} + \underbrace{\begin{bmatrix} 1 \\ \underline{\alpha}^{1} \\ \underline{\alpha}^{2} \\ \vdots \\ \underline{\alpha}^{(n-2)} \\ \underline{\alpha}^{(n-1)} \end{bmatrix}}_{\text{CCW fund. sequence}} \underline{y}_n \right\} \quad (33)$$

From (5), it is easy to see that $\underline{\alpha}^n = -1$, therefore $\underline{\alpha}^{ni} = (-1)^i$. Hence for all odd values of i, (24) will equal zero and for all even values of i, $\left(1 + \underline{\alpha}^{ni}\right) = 2$. Hence, (24) can be written as

$$\underline{y}_i = \frac{1}{2n} \sum_{k=1}^{n} 2\underline{\alpha}^{(i-1)(k-1)} \underline{x}_k \Big|_{i \in \{2,4,\ldots,2n\}}, \quad (25)$$

where one will notice that the index value i is now restricted to even values only. Adjusting the index value, (25) can be written in the following form

$$\underline{y}_i = \frac{1}{n} \sum_{k=1}^{n} \underline{\alpha}^{(2i-1)(k-1)} \underline{x}_k \Big|_{i \in \{1,2,\ldots,n\}}. \quad (26)$$

Equation (26) represents the compact generalized form of the Modified Symmetrical Component transformation. Its inverse can be shown to be

$$\underline{x}_k = \sum_{i=1}^{n} \underline{\alpha}^{-(2i-1)(k-1)} \underline{y}_i \Big|_{k \in \{1,2,\ldots,n\}}. \quad (27)$$

Equations (26) and (27) can be said to be the voltage invariant form of the transformation. The power invariant forms are given by

$$\underline{y}_i = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \underline{\alpha}^{(2i-1)(k-1)} \underline{x}_k \Big|_{i \in \{1,2,\ldots,n\}} \quad (28)$$

$$\underline{x}_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underline{\alpha}^{-(2i-1)(k-1)} \underline{y}_i \Big|_{k \in \{1,2,\ldots,n\}}. \quad (29)$$

The reader may note that the inverse of the power invariant form is simply the conjugate transpose of the forward transformation. In matrix form

$$\boldsymbol{x}_{123} = [\mathbf{F}]\, \boldsymbol{y}_{123} \quad (30)$$

$$\boldsymbol{y}_{123} = \left([\mathbf{F}]^*\right)^T x_{123} \quad (31)$$

where, from (29), the matrix $[\mathbf{F}]$ can be written as

$$[\mathbf{F}] = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \underline{\alpha}^{-1} & \underline{\alpha}^{-3} & \cdots & \underline{\alpha}^{3} & \underline{\alpha}^{1} \\ \underline{\alpha}^{-2} & \underline{\alpha}^{-6} & \cdots & \underline{\alpha}^{6} & \underline{\alpha}^{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \underline{\alpha}^{-(n-2)} & \underline{\alpha}^{-(n-6)} & \cdots & \underline{\alpha}^{(n-6)} & \underline{\alpha}^{(n-2)} \\ \underline{\alpha}^{-(n-1)} & \underline{\alpha}^{-(n-3)} & \cdots & \underline{\alpha}^{(n-3)} & \underline{\alpha}^{(n-1)} \end{bmatrix}. \quad (32)$$

Equations (30) and (32) can be parsed into the vector summation given in (33) from which one can more easily discern the significance of the individual symmetrical components.

It is easy to see that $\underline{y}_1$ and $\underline{y}_n$ represent the fundamental sequence components, the former is applied in a clock-wise direction while the latter is applied in the counter clock-wise direction. Hence if the phase vectors represented time variables, the component $\underline{y}_1$ would be considered the fundamental positive sequence component (as each subsequent vector lags the preceding vector by $\frac{\pi}{n}$ radians) and $\underline{y}_n$ would represent the fundamental negative sequence component. Similarly $\underline{y}_2$ represents the CW third-harmonic sequence. If $n = 3$, the elements of 3$^{\text{rd}}$ harmonic seq are all real, as is always the case for the zero sequence component. The component $\underline{y}_3$ represents the CW fifth-harmonic sequence. In a three-phase machine, the fifth-harmonic sequence is the negative fundamental sequence. And indeed for $n = 3$, $\left[1, \underline{\alpha}^{-5}, \underline{\alpha}^{-10}\right]^T = \left[1, \underline{\alpha}^{1}, \underline{\alpha}^{2}\right]^T$, which is the same as the CCW fundamental sequence.

For any value of n, the mathematical description of the symmetrical components does not change. Only the sequence that the component represents changes. In the three-phase machine y_3, which represents the fifth harmonic sequence, is the negative fundamental sequence, whereas for the five-phase machine, the fifth harmonic sequence represents the zero-sequence component.

C. Practical Application of the Pole-symmetric Transformation

Consider now the case of the practical three phase machine. The practical winding configuration matrix $[\mathbf{P}]$ is given in (8). It has been stated before that $[\mathbf{P}]^{-1} = [\mathbf{P}]^T$ and $[\mathbf{F}]^{-1} =$

$\left([\mathbf{F}]^*\right)^T$. Then, from (11), (31) and (32) and making use of (20), the symmetrical components for the practical three-phase machine are given by

$$\begin{aligned}\boldsymbol{y}_{123} &= \left([\mathbf{F}]^*\right)^T [\mathbf{P}]^T \boldsymbol{x}_{abc} \\ &= \frac{1}{\sqrt{3}}\begin{bmatrix}1 & \underline{\alpha}^1 & \underline{\alpha}^2 \\ 1 & \underline{\alpha}^3 & \underline{\alpha}^6 \\ 1 & \underline{\alpha}^{-1} & \underline{\alpha}^{-2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}\boldsymbol{x}_{abc} \qquad (34) \\ &= \frac{1}{\sqrt{3}}\begin{bmatrix}1 & \underline{\alpha}^2 & \underline{\alpha}^4 \\ 1 & 1 & 1 \\ 1 & \underline{\alpha}^4 & \underline{\alpha}^2\end{bmatrix}\boldsymbol{x}_{abc}.\end{aligned}$$

Then, making use of the fact that $\underline{\alpha}^{2k} = \underline{a}^k$, (34) can be written as

$$\boldsymbol{y}_{123} = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & \underline{a} & \underline{a}^2 \\ 1 & 1 & 1 \\ 1 & \underline{a}^2 & \underline{a}\end{bmatrix}. \qquad (35)$$

The reader should note that this result using the modified symmetrical component transformation, together with the winding configuration matrix, yields the same result as that of the original method of symmetrical components. The only difference is in the order in which the elements of the symmetrical component vector appear $\left(\text{e.g. } \boldsymbol{y}_{123} = \begin{bmatrix}\underline{y}_+ & \underline{y}_0 & \underline{y}_-\end{bmatrix}^T\right)$ and the fact that (35) is given in the power invariant form.

IV. Generalized Clarke and Park Transformations

The generalized Clarke and Park transformations follow directly from the modified symmetrical component transformation.

A. Generalized Clarke Transformation

The method of symmetrical components uses a complex transformation to act on complex phasors. But Edith Clarke was interested in applying the method to dynamic systems; she wondered if it could be used in the case where the elements of the terminal vector (e.g. x_{abc} or x_{123}) were instantaneous–or real–values [18]. She noticed that when one restricts the values to be real, the symmetrical components are always complex conjugates of one another

$$\underline{y}_i = \underline{y}^*_{[n-(i-1)]} \qquad (36)$$

except in the case that n is odd. In that case, there is one component (the zero sequence component), that is always real. Given this fact, she devised a transformation that would yield the real and imaginary values (α and β, respectively) of the resulting symmetrical components

$$\begin{bmatrix}x_{\alpha 1} \\ x_{\beta 1} \\ x_{\alpha 3} \\ x_{\beta 3} \\ \vdots \\ x_{\alpha n}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0 & \dots & 0 & \dots & 0 & 1 \\ -j & 0 & \dots & 0 & \dots & 0 & j \\ 0 & 1 & \dots & 0 & \dots & 1 & 0 \\ 0 & -j & \dots & 0 & \dots & j & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & \dots & 0 & 0\end{bmatrix}\begin{bmatrix}\underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_{(n+1)/2} \\ \vdots \\ \underline{y}_{(n-1)} \\ \underline{y}_n\end{bmatrix}, \qquad (37)$$

where the center column and the center element of the symmetrical component vector, $\underline{y}_{(n+1)/2}$, would only exist in the case that the phase order n is odd. Applying (37) to (31), making use of (32) and simplifying, the generalized Clarke transformation is derived as

$$\boldsymbol{x}_{123} = [\mathbf{C}]\, \boldsymbol{x}_{\alpha\beta} \qquad (38)$$

where

$$[\mathbf{C}] = \sqrt{\frac{2}{n}}\begin{bmatrix}1 & 0 & 1 & \dots & \frac{1}{\sqrt{2}} \\ \cos\delta & \sin\delta & \cos 3\delta & \dots & \frac{-1}{\sqrt{2}} \\ \cos 2\delta & \sin 2\delta & \cos 6\delta & \dots & \frac{1}{\sqrt{2}} \\ \cos 3\delta & \sin 3\delta & \cos 9\delta & \dots & \frac{-1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos(n-1)\delta & \sin(n-1)\delta & \cos(n-3)\delta & \dots & \frac{1}{\sqrt{2}}\end{bmatrix} \qquad (39)$$

and where

$$\delta = \frac{\pi}{n}. \qquad (40)$$

It can be shown that the generalized power invariant form of the Clarke transformation in (39) also has the property that its inverse is simply the transpose

$$[\mathbf{C}]^{-1} = [\mathbf{C}]^T \qquad (41)$$

Equations (38) through (40) define the generalized n-phase Clarke transformation for the fundamental winding configuration. Similar to (33), the column vectors of (39) make up the basis vectors of an n-dimensional vector space and $\boldsymbol{x}_{123}$ is determined by the linear combination of these basis vectors

$$\begin{aligned}\boldsymbol{x}_{123} &= \boldsymbol{u}_{\alpha_1} x_{\alpha_1} + \boldsymbol{u}_{\beta_1} x_{\beta_1} \\ &\quad + \boldsymbol{u}_{\alpha_3} x_{\alpha_3} + \boldsymbol{u}_{\beta_3} x_{\beta_3} + \ldots + \boldsymbol{u}_{\alpha_n} + x_{\alpha_n}. \qquad (42)\end{aligned}$$

It is not difficult to prove that these basis vectors are not only linearly independent, but orthogonal as well. Then applying (11) to (39), the Clarke transformation to practical terminal coordinates can be computed

$$\boldsymbol{x}_{abc} = [\mathbf{P}]\,[\mathbf{C}]\, x_{\alpha\beta}. \qquad (43)$$

The resulting basis vectors can be shown to be identical to those computed by Zhao using the method of vector space decomposition [9], [10]. The methodology reported here shows how those vector spaces are derived from the basic transformations.

$$[\mathbf{T}(\boldsymbol{\theta}_k)] = \sqrt{\frac{2}{n}} \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & \sin\theta_3 & \cos\theta_3 & \cdots & \frac{1}{\sqrt{2}} \\ \sin(\theta_1 - \delta) & \cos(\theta_1 - \delta) & \sin(\theta_3 - 3\delta) & \cos(\theta_3 - 3\delta) & \cdots & \frac{-1}{\sqrt{2}} \\ \sin(\theta_1 - 2\delta) & \cos(\theta_1 - 2\delta) & \sin(\theta_3 - 6\delta) & \cos(\theta_3 - 6\delta) & \cdots & \frac{1}{\sqrt{2}} \\ \sin(\theta_1 - 3\delta) & \cos(\theta_1 - 3\delta) & \sin(\theta_3 - 9\delta) & \cos(\theta_3 - 9\delta) & \cdots & \frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin(\theta_1 - (n-2)\delta) & \cos(\theta_1 - (n-2)\delta) & \sin(\theta_3 - 3(n-2)\delta) & \cos(\theta_3 - 3(n-2)\delta) & \cdots & -\frac{1}{\sqrt{2}} \\ \sin(\theta_1 - (n-1)\delta) & \cos(\theta_1 - (n-1)\delta) & \sin(\theta_3 - 3(n-1)\delta) & \cos(\theta_3 - 3(n-1)\delta) & \cdots & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{52}$$

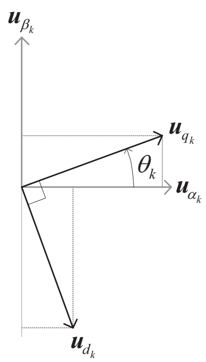


Fig. 4. Rotational basis vectors in the $\alpha_k\beta_k$ plane

In the case of the practical three-phase machine, this transformation yields

$$\begin{aligned} \boldsymbol{x}_{abc} &= \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ \cos\frac{\pi}{3} & \sin\frac{\pi}{3} & \frac{-1}{\sqrt{2}} \\ \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} & \frac{1}{\sqrt{2}} \end{bmatrix} \boldsymbol{x}_{\alpha\beta} \\ &= \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \boldsymbol{x}_{\alpha\beta} \end{aligned} \tag{44}$$

which the reader may recognize as the well-known three-phase Clarke transformation in the power invariant form. Hence, it is shown that this methodology replicates the transformations for the ubiquitous three-phase case, but is more easily extended to any number of phases.

B. Generalized Park Transformation

Park devised the rotational transform as a means to render the time-varying inductance of salient synchronous machines [19], [20]. The same can be done for higher-phase-order synchronous machines.

Suppose there are two other orthogonal basis vectors, ($\boldsymbol{u}_{d_k}$, $\boldsymbol{u}_{q_k}$) that are in the plane formed by the basis vectors ($\boldsymbol{u}_{\alpha_k}$, $\boldsymbol{u}_{\beta_k}$), but are rotated from ($\boldsymbol{u}_{\alpha_k}$, $\boldsymbol{u}_{\beta_k}$) by some arbitrary angle θ_k as depicted in Fig. 4. The relationship between the two sets of basis vectors is described by

$$\boldsymbol{u}_{d_k} = \boldsymbol{u}_{\alpha_k}\sin\theta_k - \boldsymbol{u}_{\beta_k}\cos\theta_k \tag{45}$$

$$\boldsymbol{u}_{q_k} = \boldsymbol{u}_{\alpha_k}\cos\theta_k + \boldsymbol{u}_{\beta_k}\sin\theta_k. \tag{46}$$

Furthermore, since all other $n-2$ basis vectors are orthogonal to ($\boldsymbol{u}_{\alpha_k}$, $\boldsymbol{u}_{\beta_k}$) and ($\boldsymbol{u}_{d_k}$, $\boldsymbol{u}_{q_k}$) are in the same plane, then ($\boldsymbol{u}_{d_k}$, $\boldsymbol{u}_{q_k}$) are also orthogonal to all other $n-2$ basis vectors. The vector $\boldsymbol{x}_{123}$ can be represented with respect to the new basis vectors as

$$\begin{aligned} \boldsymbol{x}_{123} = \boldsymbol{u}_{\alpha_1}x_{\alpha_1} + \boldsymbol{u}_{\beta_1}x_{\beta_1} + \ldots \\ + \boldsymbol{u}_{d_k}x_{d_k} + \boldsymbol{u}_{q_k}x_{q_k} + \ldots + \boldsymbol{u}_{\alpha_n}x_{\alpha_n} \end{aligned} \tag{47}$$

where

$$\begin{bmatrix} x_{\alpha_k} \\ x_{\beta_k} \end{bmatrix} = \begin{bmatrix} \sin\theta_k & \cos\theta_k \\ -\cos\theta_k & \sin\theta_k \end{bmatrix} \begin{bmatrix} x_{d_k} \\ x_{q_k} \end{bmatrix} \tag{48}$$

This logic can extended to each plane in the vector space. The new set of dq vectors is computed by post-multiplying the Clarke transformation by the rotational matrix

$$[\mathbf{R}(\boldsymbol{\theta}_k)] = \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & 0 & 0 & \ldots & 0 \\ -\cos\theta_1 & \sin\theta_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \sin\theta_3 & \cos\theta_3 & \ldots & 0 \\ 0 & 0 & -\cos\theta_3 & \sin\theta_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \tag{49}$$

where the final row and column will exist only if n is odd. The values of θ_1, θ_3, etc. are completely arbitrary, but θ_1 is most commonly set to align $\boldsymbol{u}_{d_1}$ with the direct axis of the rotor. In the case of a salient machine, this renders the otherwise varying inductance as a constant. In the case of higher phase order salient machines, setting value of $\theta_k = k\theta_1$ will render the inductance in the higher order planes to be constant as well. Hence, the dq transformation may be given by

$$\boldsymbol{x}_{123} = [\mathbf{T}(\boldsymbol{\theta}_k)]\,\boldsymbol{x}_{dq}, \tag{50}$$

where

$$[\mathbf{T}(\boldsymbol{\theta}_k)] = [\mathbf{C}]\,[\mathbf{R}(\boldsymbol{\theta}_k)]\,. \tag{51}$$

The matrix multiplication is carried out and the result is shown in (52).

It follows that the practical terminal values are given by

$$\boldsymbol{x}_{abc} = [\mathbf{P}]\,[\mathbf{T}(\boldsymbol{\theta}_k)]\,\boldsymbol{x}_{dq}. \tag{53}$$

It can easily be verified by the reader that the applying the configuration matrix for the practical three-phase machine yields the standard power invariant form of the three-phase dq transformation.

V. Modeling Methodology

In Sections II, III and IV the authors have introduced the concept of the fundamental winding configuration based on pole-symmetry together with the practical winding configuration matrix, have derived the generalized n-phase Symmetrical Component, Clarke and Park transformations based on the fundamental configuration and have shown how to transform from the fundamental coordinates to the practical terminal coordinates. In an upcoming paper, the authors will present the derivation of a generalized model of an n phase salient pole synchronous machine. Expressions for the stator inductance $[\mathbf{L}_{ss}(\theta_r)]$ and the stator-rotor mutual inductance $\boldsymbol{L}_{sr}(\theta_r)$ are derived based on the fundamental winding configuration, and may be transformed to practical terminal coordinates for any possible practical winding configuration.

The stator voltage $\boldsymbol{v}_{abc_s}$ can be expressed in terms of the stator current $\boldsymbol{i}_{abc_s}$ and the flux linkage $\boldsymbol{\lambda}_{abc_s}$ as

$$\boldsymbol{v}_{abc_s} = r_s \boldsymbol{i}_{abc_s} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\lambda}_{abc_s}, \tag{54}$$

where the stator flux linkage is given by

$$\boldsymbol{\lambda}_{abc_s} = [\mathbf{P}]\,[\mathbf{L}_{ss}(\theta_r)]\,[\mathbf{P}]^{-1}\,\boldsymbol{i}_{abc_s} + [\mathbf{P}]\,\boldsymbol{L}_{sr}\,(\theta_r)\,i_r. \tag{55}$$

Equation (54) may be transformed to dq coordinates by applying (53) to the stator variables

$$\begin{aligned}&[\mathbf{P}]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{v}_{dq_s} = r_s\,[\mathbf{P}]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{i}_{dq_s}\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\Big[[\mathbf{P}]\,[\mathbf{L}_{ss}(\theta_r)]\,[\mathbf{P}]^{-1}\,[\mathbf{P}]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{i}_{dq_s} + [\mathbf{P}]\,\boldsymbol{L}_{sr}\,(\theta_r)\,i_r\Big].\end{aligned} \tag{56}$$

Of course, $[\mathbf{P}]\,[\mathbf{P}]^{-1} = [\mathbf{I}]$ and since each term is pre-multiplied by $[\mathbf{P}]$, it can be brought out to the left

$$\begin{aligned}[\mathbf{P}]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{v}_{dq_s} = [\mathbf{P}]\,\big\{&r_s\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{i}_{dq_s}\\ &+\frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{L}_{ss}(\theta_r)]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{i}_{dq_s} + \boldsymbol{L}_{sr}\,(\theta_r)\,i_r\right)\Big\}.\end{aligned} \tag{57}$$

Since all of the terms inside the derivative can be time-dependent, the product rule of differentiation applies

$$\begin{aligned}[\mathbf{P}]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\boldsymbol{v}_{dq_s} &= [\mathbf{P}]\,\Big\{\Big(r_s\,[\mathbf{T}\,(\boldsymbol{\theta}_k)] + \left[\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{ss}\,(\theta_r)\right][\mathbf{T}\,(\boldsymbol{\theta}_k)]\\ &+ [\mathbf{L}_{ss}\,(\theta_r)]\left[\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}\,(\boldsymbol{\theta}_k)\right]\Big)\,\boldsymbol{i}_{dq_s} + [\mathbf{L}_{ss}\,(\theta_r)]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{i}_{dq_s}\\ &+ \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_{sr}\,(\theta_r)\,i_r + \boldsymbol{L}_{sr}\,(\theta_r)\,\tfrac{\mathrm{d}}{\mathrm{d}t}i_r\Big\}\end{aligned} \tag{58}$$

Then, multiplying through by the inverse Park transformation $[\mathbf{T}\,(\boldsymbol{\theta}_k)]^{-1}\,[\mathbf{P}]^{-1}$, the stator equation in dq coordinates becomes

$$\begin{aligned}\boldsymbol{v}_{dq_s} &= r_s\,[\mathbf{I}]\,\boldsymbol{i}_{dq_s}\\ &+ [\mathbf{T}\,(\theta_k)]^{-1}\left([\mathbf{L}_{ss}\,(\theta_r)]\,[\mathbf{T}\,(\boldsymbol{\theta}_k)]\,\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{i}_{dq_s} + \boldsymbol{L}_{sr}\,(\theta_r)\,\tfrac{\mathrm{d}}{\mathrm{d}t}i_r\right)\\ &+ [\mathbf{T}\,(\theta_k)]^{-1}\Big\{\Big(\left[\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{ss}\,(\theta_r)\right][\mathbf{T}\,(\theta_k)]\\ &+ [\mathbf{L}_{ss}\,(\theta_r)]\left[\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}\,(\boldsymbol{\theta}_k)\right]\Big)\,\boldsymbol{i}_{dq_s} + \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_{sr}\,(\theta_r)\,i_r\Big\}\end{aligned} \tag{59}$$

One will notice in (59) that nowhere in the equation does the configuration matrix appear. The dq model of the ac machine does not depend on the particular winding configuration (but the excitation does).

References

[1] E. Ward and H. Harer, "Preliminary investigation of an invertor-fed 5-phase induction motor," vol. 116, no. 6, june 1969, pp. 980 –984.

[2] R. Nelson and P. Krause, "Induction machine analysis for arbitrary displacement between multiple winding sets," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-93, no. 3, pp. 841–848, May 1974.

[3] C. Holley and D. Willyoung, "Stator winding systems with reduced vibratory forces for large turbine-generators," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-89, no. 8, pp. 1922 –1934, nov. 1970.

[4] T. M. Jahns, "Improved reliability in solid-state ac drives by means of multiple independent phase drive units," *Industry Applications, IEEE Transactions on*, vol. IA-16, no. 3, pp. 321–331, May 1980.

[5] E. Klingshirn, "High phase order induction motors - part I-description and theoretical considerations," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-102, no. 1, pp. 47–53, Jan 1983.

[6] ——, "High phase order induction motors - part II-experimental results," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-102, no. 1, p. 1, 54-59 1983.

[7] K. Pavithran, R. Parimelalagan, and M. Krishnamurthy, "Studies on inverter-fed five-phase induction motor drive," *Power Electronics, IEEE Transactions on*, vol. 3, no. 2, pp. 224–235, Apr 1988.

[8] Y. Zhao, "Vector space decomposition modeling and control of multi-phase induction machines," Ph.D. dissertation, University of Wisconsin Madison, 1995.

[9] Y. Zhao and T. Lipo, "Modeling and control of a multi-phase induction machine with structural unbalance, part I," *Energy conversion, ieee transactions on*, vol. 11, no. 3, pp. 570–577, Sep 1996.

[10] ——, "Modeling and control of a multi-phase induction machine with structural unbalance, part II," *Energy conversion, ieee transactions on*, vol. 11, no. 3, pp. 578–584, Sep 1996.

[11] H. Toliyat, S. Waikar, and T. Lipo, "Analysis and simulation of five-phase synchronous reluctance machines including third harmonic of airgap mmf," *Industry Applications, IEEE Transactions on*, vol. 34, no. 2, pp. 332 –339, mar/apr 1998.

[12] E. Semail, A. Bouscayrol, and J. Hautier, "Vectorial formalism for analysis and design of polyphase synchronous machines," *Eur. Phys. J. AP*, vol. 22, no. 3, pp. 207–220, June 2003.

[13] L. Parsa and H. Toliyat, "Fault-tolerant five-phase permanent magnet motor drives," in *Industry Applications Conference, 2004. 39th IAS Annual Meeting. Conference Record of the 2004 IEEE*, vol. 2, Oct. 2004, pp. 1048–1054 vol.2.

[14] J. Figueroa, J. Cros, and P. Viarouge, "Generalized transformations for polyphase phase-modulation motors," *Energy Conversion, IEEE Transactions on*, vol. 21, no. 2, pp. 332–341, June 2006.

[15] L. Pereira, C. Scharlau, L. Pereira, and J. Haffner, "General model of a five-phase induction machine allowing for harmonics in the air gap field," *Energy Conversion, IEEE Transactions on*, vol. 21, no. 4, pp. 891–899, Dec. 2006.

[16] D. C. White and H. H. Woodson, *Electromechanical Energy Conversion*. John Wiley & Sons, 1959.

[17] C. L. Fortescue, "Method of symmetrical co-ordinates applied to the solution of polyphase networks," *American Institute of Electrical Engineers, Transactions of the*, vol. XXXVII, no. 2, pp. 1027–1140, July 1918.

[18] E. Clarke, *Circuit Analysis of AC Power Systems–Symmetrical and Related Components*. John Wiley & Sons, 1943, vol. I.

[19] R. H. Park, "Two-reaction theory of synchronous machines generalized method of analysis-part I," *American Institute of Electrical Engineers, Transactions of the*, vol. 48, no. 3, pp. 716–727, July 1929.

[20] ——, "Two-reaction theory of synchronous machines - II," *American Institute of Electrical Engineers, Transactions of the*, vol. 52, no. 2, pp. 352–354, June 1933.